BIHARMONIC RIEMANNIAN SUBMERSIONS FROM THE PRODUCT SPACE $M^2 \times \mathbb{R}$

ZE-PING WANG* AND YE-LIN OU**

Abstract

In this paper, we study biharmonic Riemannian submersions $\pi : M^2 \times \mathbb{R} \to (N^2, h)$ from a product manifold onto a surface and obtain some local characterizations of such biharmonic maps. Our results show that when the target surface is flat, then a proper biharmonic Riemannian submersion $\pi : M^2 \times \mathbb{R} \to (N^2, h)$ is locally a projection of a special twisted product, and when the target surface is non-flat, π is locally a special map between two warped product spaces with a warping function that solves a single ODE. As a by-product, we also prove that there is a unique proper biharmonic Riemannian submersion $H^2 \times \mathbb{R} \to \mathbb{R}^2$ given by the projection of a warped product.

1. Introduction and preliminaries

A **harmonic map** is a map $\varphi : (M, g) \to (N, h)$ between Riemannian manifolds whose tension filed vanishes identically, i.e., $\tau(\varphi) = \mathrm{Trace}_g \nabla \mathrm{d}\varphi \equiv 0$. A **biharmonic map** is one whose bitension filed solves the PDEs

$$\tau_2(\varphi) := \mathrm{Trace}_g(\nabla^{\varphi}\nabla^{\varphi} - \nabla^{\varphi}_{\nabla^M})\tau(\varphi) - \mathrm{Trace}_g R^N(\mathrm{d}\varphi, \tau(\varphi))\mathrm{d}\varphi = 0, \tag{1}$$

where R^N is the curvature operator of (N, h) defined by

$$R^N(X, Y)Z = [\nabla^N_X, \nabla^N_Y]Z - \nabla^N_{[X,Y]}Z.$$

Clearly, any harmonic map is a biharmonic map. A biharmonic map which is not harmonic is called a **proper biharmonic map**.

The geometric study of biharmonic maps focuses on biharmonicity of maps with geometric interest like isometric immersions or Riemannian submersions,

Date: 2/24/2024.

1991 *Mathematics Subject Classification.* 58E20, 53C12, 53C42.

Key words and phrases. Biharmonic maps, biharmonic Riemannian submersions, product spaces.

*Supported by the Natural Science Foundation of China (No. 11861022).

** Supported by a grant from the Simons Foundation (427231, Ye-Lin Ou).

and the geometry and topology of the spaces related to the existence of such geometrically special biharmonic maps. For example, in the study of biharmonic submanifolds (biharmonic isometric immersions), a fundamental problem is to classify biharmonic submanifolds in certain model spaces. Although many progresses have been made the following conjectures remain open in general cases.

Chen's Conjecture ([8])**:** Any biharmonic submanifolds in Euclidean space is minimal.

Conjecture (Balmus-Montaldo-Oniciuc [3]**):** A biharmonic submanifold in sphere has constant mean curvature; and any proper biharmonic hypersurface in S^{m+1} is an open part of $S^m(\frac{1}{\sqrt{2}})$ or the generalized Clifford torus $S^p(\frac{1}{\sqrt{2}})\times S^q(\frac{1}{\sqrt{2}})$ with $p+q=m, p\neq q$.

For more detailed information and some recent progress on biharmonic submanifolds see the recent book [19] and the vast references therein.

Riemannian submersions are a dual concept of isometric immersions (i.e., submanifolds). The study of biharmonicity of Riemannian submersions was initiated in [16]. A useful tool of using the so-called integrability data to study biharmonic Riemannian submersion from a generic 3-manifold was introduced in [21]. This was later generalized to higher dimensions with one dimensional fibers in [1]. Complete classifications of Riemannian submersions from a 3-dimensional space forms and more general BCV spaces into a surface were obtained in [21, 22].

In this paper, we study biharmonic submersions from the product space $M^2\times\mathbb{R}$, where M^2 is a general 2-dimensional manifold. A reason for the choice of the product space $M^2\times\mathbb{R}$ is that it includes many well-known model spaces, such as $S^2\times\mathbb{R}$, $H^2\times\mathbb{R}$, $\mathbb{R}^3$, and twisted spaces $\mathbb{R}^2\times_{e^{2p}}\mathbb{R}=(\mathbb{R}^3, dy^2+dz^2+e^{2p(x,y)}dx^2)$. We give some local characterizations of biharmonic (including harmonic) Riemannians submersions from $M^2\times\mathbb{R}$ onto a surface. These include: a Riemannian submersion $\pi: M^2\times\mathbb{R}\to(N^2,h)$ is harmonic if and only if it is locally the projection onto the first factor followed by a Riemannian covering map (Theorem 2.2); A Riemannian submersion $\pi: M^2\times\mathbb{R}\to\mathbb{R}^2$ is proper biharmonic, then it is locally a projection of a special twisted product (Theorem 2.9), and if a Riemannian submersion $\pi: M^2\times\mathbb{R}\to(N^2,h)$ into a non-flat surface is proper biharmonic, then π is locally a special map between two warped product spaces with a warping function that solves a single ODE (Corollary 2.10).

2. Harmonic and biharmonic Riemannian submersions from $M^2 \times \mathbb{R}$

We will use the following useful local orthonormal frame in the paper.

Lemma 2.1. *For any point on $M^2 \times \mathbb{R}$, there is a neighborhood $U \times \mathbb{R}$ and local coordinates (t, s, z) such that the product metric on $M^2 \times \mathbb{R}$ takes the form $e^{2q(t,s)}dt^2 + ds^2 + dz^2$. Furthermore, the orthonormal frame $\{E_1 = e^{-q(t,s)}\partial_t,\ E_2 = \partial_s, E_3 = \partial_z\}$ satisfies*

$$\begin{aligned}&[E_1, E_2] = fE_1, \ \text{all other}\ [E_i, E_j] = 0,\ i, j = 1, 2, 3,\\ &\nabla_{E_1} E_1 = -fE_2,\ \nabla_{E_1} E_2 = fE_1,\ \text{all other}\ \nabla_{E_i} E_j = 0,\ i,\ j = 1, 2, 3,\end{aligned} \tag{2}$$

where $f = q_s$, and the only possible nonzero components of the (Ricci) curvatures of $M^2 \times \mathbb{R}$ are given by

$$\mathrm{Ric}\,(E_1, E_1) = \mathrm{Ric}(E_2, E_2) = R_{1212} = g(R(E_1, E_2)E_2, E_1) = -E_2(f) - f^2, \tag{3}$$

which is the Gauss curvature of M^2 on U denoted by $K^{M^2} = -E_2(f) - f^2$.

Proof. It is well known that around any point of a 2-dimensional Riemannian manifold (M^2, g) there exists a semi-geodesic coordinate system so that locally it can be described as $(U \subseteq M^2, g = e^{2q(t,s)}dt^2 + ds^2)$. One can easily checked that $\{E_1 = e^{-q(t,s)}\partial_t,\ E_2 = \partial_s, E_3 = \partial_z\}$ is an orthonormal frame satisfies (2) and (3) with $f = q_s$. □

Remark 1. Note that the Lemma 2.1 actually implies that any local semi-geodesic coordinate system gives rise to an orthonormal frame $\{E_i\}$ with the properties (2) and (3). For latter reference we call such an orthonormal frame *semi-geodesic orthonormal frame.*

Let $\pi : M^2 \times \mathbb{R} \to (N^2, h)$ be a Riemannian submersion, and let $\{e_1,\ e_2,\ e_3\}$ be a local orthonormal frame on $W \subset M^2 \times \mathbb{R}$ with e_3 vertical. Let $\{f_1,\ f_2, f_3, \kappa_1,\ \kappa_2,\ \sigma\}$ be the generalized integrability associated to this frame. Note that $f_3 = 0$ if and only if the above frame is adapted to π.

Using the relation $e_i = \sum_{j=1}^{3} a_i^j E_j$, as it was computed in [22], we have the following Lie brackets and the components of the Levi-Civita connection with respect to this frame as

$$\begin{aligned}&[e_1, e_3] = f_3e_2 + \kappa_1e_3,\ [e_2, e_3] = -f_3e_1 + \kappa_2e_3,\ [e_1, e_2] = f_1e_1 + f_2e_2 - 2\sigma e_3,\\ &\nabla_{e_1} e_1 = -f_1e_2,\ \ \nabla_{e_1} e_2 = f_1e_1 - \sigma e_3,\ \ \nabla_{e_1} e_3 = \sigma e_2,\\ &\nabla_{e_2} e_1 = -f_2e_2 + \sigma e_3,\ \ \nabla_{e_2} e_2 = f_2e_1,\ \ \nabla_{e_2} e_3 = -\sigma e_1,\\ &\nabla_{e_3} e_1 = -\kappa_1e_3 + (\sigma - f_3)e_2, \nabla_{e_3} e_2 = -(\sigma - f_3)e_1 - \kappa_2e_3, \nabla_{e_3} e_3 = \kappa_1e_1 + \kappa_2e_2,\end{aligned} \tag{4}$$

and the only possible nonzero components of the Riemannian curvature R of $M^2\times\mathbb{R}$ as

$$(5)\quad \begin{cases} R(e_1,e_3,e_1,e_2)=-e_1(\sigma)+2\kappa_1\sigma=-a_2^3a_3^3K^{M^2},\\ R(e_1,e_3,e_1,e_3)=e_1(\kappa_1)+\sigma^2-\kappa_1^2+\kappa_2f_1=(a_2^3)^2K^{M^2},\\ R(e_1,e_3,e_2,e_3)=e_1(\kappa_2)-e_3(\sigma)-\kappa_1f_1-\kappa_1\kappa_2=-a_1^3a_2^3K^{M^2},\\ R(e_1,e_2,e_1,e_2)=e_1(f_2)-e_2(f_1)-f_1^2-f_2^2+2f_3\sigma-3\sigma^2=(a_3^3)^2K^{M^2},\\ R(e_1,e_2,e_2,e_3)=-e_2(\sigma)+2\kappa_2\sigma=a_1^3a_3^3K^{M^2},\\ R(e_2,e_3,e_1,e_3)=e_2(\kappa_1)+e_3(\sigma)+\kappa_2f_2-\kappa_1\kappa_2=-a_1^3a_2^3K^{M^2},\\ R(e_2,e_3,e_2,e_3)=\sigma^2+e_2(\kappa_2)-\kappa_1f_2-\kappa_2^2=(a_1^3)^2K^{M^2}. \end{cases}$$

Finally, the Gauss curvature of the base space is given by

$$(6)\qquad K^N=e_1(f_2)-e_2(f_1)-f_1^2-f_2^2+2f_3\sigma.$$

Remark 2. From (4) we see that $\sigma=0$ on the neighborhood W if and only if the horizontal distribution of the Riemannian submersion π on W is integrable.

2.1. **Harmonic Riemannian submersions from $M^2\times\mathbb{R}$.** Harmonic Riemannian submersions $M^3\to N^2$ are a special subclass of horizontally homothetic harmonic morphisms with totally geodesic fibers. For general harmonic morphisms and their applications and interesting links to other areas of mathematics see the book [4]. It is well known that the only Riemannian submersion from S^3 is the Hopf fibration $S^3\to S^2$ which is harmonic since the fibers are totally geodesic. Using the Bernstein theorem for harmonic morphism (see Baird-Wood [5]), one can easily deduce (see also [21]) that there is no harmonic Riemannian submersion $H^3\to(N^2,h)$ from a 3-dimensional hyperbolic space form, and that any globally defined harmonic Riemannian submersion $\phi:\mathbb{R}^3\to(N^2,h)$ is an orthogonal projection $\mathbb{R}^3\to\mathbb{R}^2$ followed by a Riemannian covering map $\mathbb{R}^2\to(N^2,h)$. For some recent work on the classifications of harmonic Riemannian submersions from Thurston's 3-dimensional geometries, BCV 3-spaces, and Berger 3-spheres see [23]. In this subsection, we study harmonic Riemannian submersions from $M^2\times\mathbb{R}$.

Theorem 2.2. *A Riemannian submersion $\pi:M^2\times\mathbb{R}\to(N^2,h)$ is harmonic if and only if locally it is, up to an isometry of the domain, the projection onto the first factor followed by a Riemannian covering map.*

Proof. By Lemma 2.1, for any point on $M^2\times\mathbb{R}$, there is a neighborhood $W=U\times\mathbb{R}$ on which we have an orthonormal frame $\{E_i\}$. Let $e_i=\sum_{j=1}^{3}a_i^jE_j$ be a local orthonormal frame with e_3 vertical. Using Proposition 2.2 in [23] and (5), we conclude that if the Riemannian submersion $\pi : M^2 \times \mathbb{R} \supseteq W \to (N^2, h)$ is harmonic, then we have

$$\sigma^2 = (a_2^3)^2 K^{M^2} = (a_1^3)^2 K^{M^2},\ a_1^3 a_2^3 K^{M^2} = 0,\ \text{and } K^N = 3\sigma^2 + (a_3^3)^2 K^{M^2}. \tag{7}$$

By continuity of K^{M^2}, we may assume that either $K^{M^2} \equiv 0$ or $K^{M^2} \neq 0$ on a neighborhood denoted by W by an abuse of notation. For the first case, (7) implies that $\sigma = 0$, and $K^N = K^{M^2} = 0$ on W. For the second case, we use $K^{M^2} \neq 0$ on W together with the 1st and the 2nd equation of (7) to conclude that $a_1^3 = a_2^3 = 0$ and hence $(a_3^3)^2 = 1$, which also implies that $\sigma = 0$ and hence the Gauss curvature of the base space $K^N = \sigma^2 + (a_3^3)^2 K^{M^2} = (a_3^3)^2 K^{M^2} = K^{M^2}$. Thus, in either case, we have $\sigma = 0$ on a neighborhood W which means the Riemannian submersion $\pi : M^2 \times \mathbb{R} \supseteq W \to (N^2, h)$ has integrable horizontal distribution. This, together with the assumption that $\pi|_W$ is harmonic and hence it has totally geodesic fibers, implies that $\pi|_W$ is a Riemannian submersion with totally geodesic fibers and integrable horizontal distribution whose integral submanifold is an open neighborhood U of M^2 that is isometric to an open neighborhood of (N^2, h). Therefore, $\pi|_W$ is the projection along the fibers onto U followed by a Riemannian covering map onto $V \subset N^2$. □

2.2. **Biharmonic Riemannian submersions from $M^2 \times \mathbb{R}$.** In this subsection, we will characterize all proper biharmonic Riemannian submersions from $M^2 \times \mathbb{R}$, which are not harmonic.

We will use the following lemma in the rest of the paper.

Lemma 2.3. *([21]) Let $\pi : (M^3, g) \to (N^2, h)$ be a Riemannian submersion with an adapted frame $\{e_1,\ e_2,\ e_3\}$ and the integrability data $f_1, f_2, \kappa_1,\ \kappa_2$ and σ. Then, the Riemannian submersion π is biharmonic if and only if*

$$\begin{cases} -\Delta^M \kappa_1 - 2\sum_{i=1}^{2} f_i e_i(\kappa_2) - \kappa_2 \sum_{i=1}^{2} (e_i(f_i) - \kappa_i f_i) + \kappa_1 \left(-K^N + \sum_{i=1}^{2} f_i^2 \right) = 0, \\ -\Delta^M \kappa_2 + 2\sum_{i=1}^{2} f_i e_i(\kappa_1) + \kappa_1 \sum_{i=1}^{2} (e_i(f_i) - \kappa_i f_i) + \kappa_2 \left(-K^N + \sum_{i=1}^{2} f_i^2 \right) = 0, \end{cases} \tag{8}$$

where $K^N = R^N_{1212} \circ \pi = e_1(f_2) - e_2(f_1) - f_1^2 - f_2^2$ is the Gauss curvature of (N^2, h).

Lemma 2.4. *(see [22]) Let $\pi : (M^3, g) \to (N^2, h)$ be a Riemannian submersion from Riemannian 3-manifolds and $\{e_1, e_2, e_3\}$ be any local orthonormal frame with e_3 tangent to the fibers. If $\nabla_{e_1} e_1 = 0$, then either $\nabla_{e_2} e_2 = 0$; or $\nabla_{e_2} e_2 \not\equiv 0$, and the frame $\{e_1, e_2, e_3\}$ is adapted to the Riemannian submersion π.*

The main tool used to prove our main theorems is a special orthonormal frame adapted to the Riemannian submersion. First we prove the existence of such frame.

Proposition 2.5. *Let $\pi : M^2 \times \mathbb{R} \to (N^2, h)$ be a Riemannian submersion and $\{E_1,\ E_2,\ E_3 = \frac{\partial}{\partial z}\}$ be a local semi-geodesic orthonormal frame stated in Lemma 2.1. Then, there exists an orthonormal frame*

$$
\begin{cases} e_1 = \cos\theta E_1 + \sin\theta E_2, \\ e_2 = -\cos\alpha(\sin\theta E_1 - \cos\theta E_2) + \sin\alpha E_3, \\ e_3 = \sin\alpha(\sin\theta E_1 - \cos\theta E_2) + \cos\alpha E_3, \end{cases} \tag{9}
$$

such that e_3 is vertical, $\nabla_{e_1} e_1 = 0$, where $\cos\theta = \langle e_1, E_1\rangle$, $\cos\alpha = \langle e_3, E_3\rangle$. Furthermore, the generalized integrability data of $\{e_i\}$ are given by

$$
\begin{aligned} & f_1 = 0,\ f_2 = -\bar{f}\cos^2\alpha - \sin\alpha\cos\alpha E_3(\theta),\ f_3 = -E_3(\theta), \\ & \sigma = -\bar{f}\sin\alpha\cos\alpha - \sin^2\alpha E_3(\theta),\ \kappa_1 = -\bar{f}\sin^2\alpha + \sin\alpha\cos\alpha E_3(\theta), \\ & \kappa_2 = \sin\alpha e_3(\cos\alpha) - \cos\alpha e_3(\sin\alpha) = -e_3(\alpha), \end{aligned} \tag{10}
$$

where

$$
\bar{f} = -\sin\theta E_1(\theta) + \cos\theta E_2(\theta) + f\sin\theta. \tag{11}
$$

Proof. Let e_3 be the unit vector field tangent to the fibers of π. Clearly, if $e_3 = \pm E_3$, then $e_1 = E_2, e_2 = -E_1, e_3 = E_3$ is such an orthonormal frame. Hereafter, we suppose that $e_3 \neq \pm E_3 = \pm\frac{\partial}{\partial z}$. In this case, we take $e_1 = \frac{e_3\times E_3}{|e_3\times E_3|}$ which is horizontal since $\langle e_1, e_3\rangle = 0$, and obtain a natural orthonormal frame $\{e_1,\ e_2 = e_3\times e_1,\ e_3\}$ on $M^2\times\mathbb{R}$ which can be expressed as

$$
e_i = \sum_{j=1}^{3} a_i^j E_j, i = 1, 2, 3, (a_i^j) \in SO(3). \tag{12}
$$

By the choices e_i, we have

$$
a_1^3 = \langle e_1, E_3\rangle = 0,\ a_3^3 \neq \pm 1,\ \text{(and hence)}\ a_2^3 \neq 0, \tag{13}
$$

and hence $e_1 = a_1^1 E_1 + a_1^2 E_2$ with $(a_1^1)^2 + (a_1^2)^2 = 1$.
Furthermore, we can check that

$$
f_1 = 0,\ \nabla_{e_1} e_1 = 0. \tag{14}
$$

In fact, a straightforward computation using (4) gives
(15)

$$
-f_1\sum_{i=1}^{3} a_2^i E_i = -f_1 e_2 = \nabla_{e_1} e_1 = \nabla_{e_1}(\sum_{i=1}^{3} a_1^i E_i) = \sum_{i=1}^{3} e_1(a_1^i)E_i + \sum_{i,j=1}^{3} a_1^j a_1^i \nabla_{E_j} E_i.
$$

Using (2), the fact that $a_1^3 = 0$, and comparing the coefficient of E_3 on both sides of (15) yields $-f_1 a_2^3 = e_1(a_1^3) = 0$. From this we obtain $f_1 = 0$ since $a_2^3 \neq 0$, and hence $\nabla_{e_1} e_1 = 0$.

It is not difficult to check that

$$\bar{E}_1 = a_1^2 E_1 - a_1^1 E_2,\ \bar{E}_2 = a_1^1 E_1 + a_1^2 E_2,\ \bar{E}_3 = E_3 \tag{16}$$

defines another orthonormal frame on $U \times \mathbb{R} \subseteq M^2 \times \mathbb{R}$ which satisfies

$$\begin{aligned} &[\bar{E}_1, \bar{E}_3] = \bar{f}_3 \bar{E}_2,\ [\bar{E}_2, \bar{E}_3] = -\bar{f}_3 \bar{E}_1,\ \ [\bar{E}_1, \bar{E}_2] = \bar{f} \bar{E}_1,\\ &\nabla_{\bar{E}_1} \bar{E}_1 = -\bar{f} \bar{E}_2,\ \nabla_{\bar{E}_1} \bar{E}_2 = \bar{f} \bar{E}_1,\ \nabla_{\bar{E}_3} \bar{E}_1 = -\bar{f}_3 \bar{E}_2,\ \nabla_{\bar{E}_3} \bar{E}_2 = \bar{f}_3 \bar{E}_1,\\ &\text{all other } \nabla_{\bar{E}_i} \bar{E}_j = 0,\ i,\ j = 1, 2, 3, \end{aligned} \tag{17}$$

with

$$\bar{f} = E_1(a_1^1) + E_2(a_1^2) + f a_1^2,\ \bar{f}_3 = a_1^2 E_3(a_1^1) - a_1^1 E_3(a_1^2), \tag{18}$$

and the only possible nonzero components of the (Ricci) curvatures given by

$$\mathrm{Ric}\,(\bar{E}_1, \bar{E}_1) = \mathrm{Ric}(\bar{E}_2, \bar{E}_2) = R_{1212} = g(R(\bar{E}_1, \bar{E}_2)\bar{E}_2, \bar{E}_1) = -\bar{E}_2(\bar{f}) - \bar{f}^2,$$

which is the Gauss curvature of M^2 on U denoted by $K^{M^2} = -\bar{E}_2(\bar{f}) - \bar{f}^2$.

We can also check that by introducing the new variables θ, α so that

$$a_1^1 = \cos\theta, a_1^2 = \sin\theta, a_2^3 = \sin\alpha,\ \ a_3^3 = \cos\alpha, \tag{19}$$

we can rewrite (16) and (12), respectively, as

$$\bar{E}_1 = \sin\theta E_1 - \cos\theta E_2,\ \bar{E}_2 = \cos\theta E_1 + \sin\theta E_2,\ \ \bar{E}_3 = E_3, \tag{20}$$

$$\begin{cases} e_1 = \cos\theta E_1 + \sin\theta E_2,\\ e_2 = -\cos\alpha(\sin\theta E_1 - \cos\theta E_2) + \sin\alpha E_3,\\ e_3 = \sin\alpha(\sin\theta E_1 - \cos\theta E_2) + \cos\alpha E_3, \end{cases} \tag{21}$$

where

$$\begin{cases} a_1^1 = \cos\theta,\ a_1^2 = \sin\theta,\ a_1^3 = 0,\\ a_2^1 = -\cos\alpha\sin\theta,\ a_2^2 = \cos\alpha\cos\theta,\ a_2^3 = \sin\alpha,\\ a_3^1 = \sin\alpha\sin\theta,\ a_3^2 = -\sin\alpha\cos\theta,\ a_3^3 = \cos\alpha. \end{cases} \tag{22}$$

It is also clear that the relationship between the orthonormal frames $\{e_i\}$ and $\{\bar{E}_i\}$ is given by

$$e_1 = \bar{E}_2,\ e_2 = -\cos\alpha\bar{E}_1 + \sin\alpha E_3,\ e_3 = \sin\alpha\bar{E}_1 + \cos\alpha E_3. \tag{23}$$

To compute the integrability data of the frame $\{e_1,\ e_2,\ e_3\}$, we first use (2), (4), $a_1^3 = f_1 = 0$, and a computation similar to those used to obtain (15), to have

$$\begin{aligned}&e_1(a_2^3) = -\sigma a_3^3,\ e_1(a_3^3) = \sigma a_2^3,\ e_2(a_2^3) = 0,\ e_2(a_3^3) = 0,\\ &e_3(a_2^3) = -\kappa_2 a_3^3,\ e_3(a_3^3) = \kappa_2 a_2^3,\ \kappa_1 a_3^3 = (\sigma - f_3)a_2^3,\ f_2 a_2^3 = \sigma a_3^3,\\ &e_1(a_1^1) = -a_1^1 a_1^2 f,\ e_1(a_1^2) = (a_1^1)^2 f.\end{aligned} \tag{24}$$

Using (2), (23), (17), the 1st and the 2nd equations of (24) we have

$$\begin{aligned}&f_3 e_2 + \kappa_1 e_3 = [e_1, e_3] = [e_1, \sin\alpha \bar{E}_1 + \cos\alpha E_3]\\ &= (\sigma + \bar{f}\sin\alpha\cos\alpha - \cos^2\alpha E_3(\theta))e_2 + (-\bar{f}\sin^2\alpha + \sin\alpha\cos\alpha E_3(\theta))e_3.\end{aligned} \tag{25}$$

Comparing coefficients on both sides of (25) yields

$$f_3 = \sigma + \bar{f}\sin\alpha\cos\alpha - \cos^2\alpha E_3(\theta),\ \kappa_1 = -\bar{f}\sin^2\alpha + \sin\alpha\cos\alpha E_3(\theta). \tag{26}$$

Similarly, by computing

$$\begin{aligned}&f_2 e_2 - 2\sigma e_3 = [e_1, e_2] = [e_1, -\cos\alpha \bar{E}_1 + \sin\alpha E_3]\\ &= (-\bar{f}\cos^2\alpha - \sin\alpha\cos\alpha E_3(\theta))e_2 + (-\sigma + \bar{f}\sin\alpha\cos\alpha + \sin^2\alpha E_3(\theta))e_3,\end{aligned} \tag{27}$$

and comparing coefficients on both sides of this equation, we have

$$\sigma = -\bar{f}\sin\alpha\cos\alpha - \sin^2\alpha E_3(\theta),\ f_2 = -\bar{f}\cos^2\alpha - \sin\alpha\cos\alpha E_3(\theta). \tag{28}$$

Using the 1st equation of (26) and the 1st equation of (28), we obtain

$$f_3 = -E_3(\theta) = \sin\theta E_3(\cos\theta) - \cos\theta E_3(\sin\theta) = a_1^2 E_3(a_1^1) - a_1^1 E_3(a_1^2). \tag{29}$$

Using the 5th and 6th equation of (24), a simple computation yields

$$\kappa_2 = \sin\alpha e_3(\cos\alpha) - \cos\alpha e_3(\sin\alpha) = -e_3(\alpha). \tag{30}$$

From (14), (26), (28), (29), and (30), we obtain the integrability data (10). □

Now we are ready to prove the following theorem which provides the main tool to prove our classification theorems.

Theorem 2.6. *Let $\pi : M^2 \times \mathbb{R} \to (N^2, h)$ be a Riemannian submersion. Then, the orthonormal frame $\{e_i\}$ given by (9) is adapted to the Riemannian submersion π with the integrability data*

$$f_1 = 0,\ f_2 = -\bar{f}\cos^2\alpha,\ f_3 = 0,\ \sigma = -\bar{f}\sin\alpha\cos\alpha,\ \kappa_1 = -\bar{f}\sin^2\alpha,\ \kappa_2 = -e_3(\alpha). \tag{31}$$

Proof. By Proposition 2.5, we have $\nabla_{e_1} e_1 = 0$. Thus, we can use Lemma 2.4 to conclude that either $\nabla_{e_2} e_2 \not\equiv 0$ in which case the chosen frame $\{e_1, e_2, e_3\}$ is adapted to the Riemannian submersion π, or $\nabla_{e_2} e_2 = 0$. So, it suffices to prove that the orthonormal frame is also adapted in the the latter case: $\nabla_{e_2} e_2 = 0$, i.e., $f_2 = 0$. Using the assumptions (13), $f_2 = 0$, the 2nd and the 8th equations of (24) we obtain $\sigma = 0$. Then, using the 4th equation of (5) with $f_1 = f_2 = \sigma = a_1^3 = 0$, we have either $K^{M^2} = 0$, or $a_3^3 = 0$ and hence $(a_2^3)^2 = 1$.

For the case of $a_3^3 = 0$ and $(a_2^3)^2 = 1$, one applies the 7th equation of (24) with $\sigma = 0$ to conclude that $f_3 = 0$, which implies that the chosen frame $\{e_1 = a_1^1 E_1 + a_1^2 E_2, e_2, e_3\}$ is adapted to the Riemannian submersion.

For the case $K^{M^2} = 0$ and $a_3^3 \neq 0, \pm 1$ (and hence $a_2^3 \neq 0, \pm 1$), we will prove that $f_3 = 0$ and hence the orthonormal frame $\{e_i\}$ is adapted to π.

First, we note that the orthonormal frame $\{\bar{E}_i\}$ given in (16) is a natural orthonormal frame with respect to the harmonic Riemannian submersion

$$\pi_1 : M^2 \times \mathbb{R} \to M^2, \ \pi_1(p, z) = p.$$

By comparing the equations on the first line of (17) with those of (4) and using (29) and the second equation of (18), we find the generalized integrability data of $\{\bar{E}_i\}$ to be

$$\bar{f}_1 = \bar{f}, \ \bar{f}_2 = \bar{\kappa}_1 = \bar{\kappa}_2 = \bar{\sigma} = 0, \ \bar{f}_3 = f_3 = -E_3(\theta). \tag{32}$$

Applying Lemma 2.4 with $e_1 = \bar{E}_2, e_2 = \bar{E}_1$, we deduce that either $\bar{f}_1 = \bar{f} \not\equiv 0$ in which case the orthonormal frame $\{\bar{E}_i\}$ is adapted to the Riemannian submersion π_1, and hence $\bar{f}_3 = f_3 = 0$, or $\bar{f}_1 = \bar{f} = 0$. For the latter case, we use the assumptions $\bar{f}_1 = \bar{f} = 0$, together with $a_3^3 \neq 0, \pm 1$ (and hence $a_2^3 \neq 0, \pm 1$), $\sigma = f_2 = 0$ and (28) to have $f_3 = -E_3(\theta) = 0$. Thus, in any case, the frame $\{e_i\}$ is adapted to the Riemannian submersion π, and the generalized integrability data (10) reduces to the integrability data (31). □

Corollary 2.7. *A biharmonic Riemannian submersion $\pi : M^2 \times \mathbb{R} \to (N^2, h)$ with $K^{M^2} = K^N = 0$ has to be harmonic.*

Proof. By Theorem 2.6, the orthonormal frame $\{e_i\}$ is adapted to the Riemannian submersion π with the integrability data (31). Using the 4th equation of (5) and the assumption that $K^{M^2} = K^N = 0$, we have $\sigma = 0$. From this and the 4th equation of (31), we have either $\bar{f} = 0$ or $\sin\alpha\cos\alpha = 0$. Therefore, we can obtain our corollary by the following cases:

Case I: $\bar{f} = 0$ and $a_2^3 a_3^3 = \sin\alpha\cos\alpha \neq 0$. Combining these, (31), and the assumption that $K^{M^2} = K^N = 0$ we have

$$a_1^1 = f_1 = f_2 = f_3 = \sigma = \kappa_1 = K^{M^2} = K^N = 0, \ a_2^3, \ a_3^3 \neq 0, \pm 1, \ \kappa_2 = -e_3(\alpha). \tag{33}$$

Using (33) and (5) we obtain

$$e_2(\kappa_2) = \kappa_2^2,\ e_1(\kappa_2) = 0. \tag{34}$$

Thus, in this case, if π is biharmonic with $K^N = 0$, then we have (4), (33), (34), and the biharmonic equation (8) reduces to

$$\Delta\kappa_2 = 0, \tag{35}$$

and hence

$$e_3e_3(\kappa_2) = -\kappa_2^3. \tag{36}$$

Applying e_2 to both sides of (36) we have

$$e_2e_3e_3(\kappa_2) = -3\kappa_2^4. \tag{37}$$

On the other hand, we use (4) and $e_2e_3(\kappa_2) - e_3e_2(\kappa_2) = [e_2, e_3](\kappa_2) = \kappa_2e_3(\kappa_2)$ to obtain

$$e_2e_3(\kappa_2) = 3\kappa_2e_3(\kappa_2). \tag{38}$$

By applying e_3 to both sides of (38) and using (36) we get

$$e_3e_2e_3(\kappa_2) = 3(e_3(\kappa_2))^2 - 3\kappa_2^4. \tag{39}$$

A further computation using (37) and (39) gives

$$\begin{aligned} &\kappa_2e_3(e_3(\kappa_2)) = [e_2, e_3](e_3(\kappa_2)) \\ &= e_2e_3e_3(\kappa_2) - e_3e_2e_3(\kappa_2) = -3(e_3(\kappa_2))^2. \end{aligned} \tag{40}$$

This, together with (36), implies

$$3(e_3(\kappa_2))^2 = \kappa_2^4, \tag{41}$$

Applying e_3 to both sides of (41) and using (36) we get

$$10\kappa_2^3e_3(\kappa_2) = 0, \tag{42}$$

which implies either $\kappa_2 = 0$ or $e_3(\kappa_2) = 0$. For the latter case, we use (41) to obtain $\kappa_2 = 0$. So in any case, the Riemannian submersion π is harmonic since $\kappa_1 = \kappa_2 = 0$.

Case II: $a_3^3 = \cos\alpha = 0$ and $a_2^3 = \sin\alpha = \pm 1$. In this case, (31) reduces to

$$f_1 = f_2 = f_3 = \sigma = \kappa_2 = 0,\ \kappa_1 = -\bar{f}. \tag{43}$$

Using these and (5) we have

$$e_1(\kappa_1) = \kappa_1^2,\ e_2(\kappa_1) = 0. \tag{44}$$

It follows that in this case, we can use (4), (43), (44) and $K^N = 0$ to conclude that the biharmonic equation (8) reduces to

$$\Delta\kappa_1 = 0. \tag{45}$$

We make a computation similar to those used to compute (36)-(42) in Case I to have $\kappa_1 = 0$. So π is harmonic since $\kappa_1 = \kappa_2 = 0$.

Case III: $a_3^3 = \cos\alpha = \pm 1$ and $\sin\alpha = 0$. In this case, using the 5th and 6th equations of (31) we have $\kappa_1 = \kappa_2 = 0$ implying that the Riemannian submersion π is harmonic.

Summarizing all the above cases we obtain the corollary.

□

Now we are ready to give a first characterization of proper biharmonic Riemannian submersions from $M^2 \times \mathbb{R}$.

Theorem 2.8. *Let $\pi : M^2 \times \mathbb{R} \to (N^2, h)$ be a proper biharmonic Riemannian submersion. Then we have*
(a) The target surface is flat, and the adapted frame (9) has the integrability data $f_1 = f_2 = \kappa_2 = \sigma = 0,\ \kappa_1 = -\bar{f} \neq 0$ solving the following PDE

$$\Delta\kappa_1 = 0, \text{ i.e., } \Delta^{M^2}\bar{f} = 0, \tag{46}$$

where Δ^{M^2} denotes the Laplacian of M^2 and $\bar{f} = -\sin\theta E_1(\theta) + \cos\theta E_2(\theta) + f\sin\theta$ is a function on $U \subseteq M^2$, or,
(b) The target surface is non-flat, and the adapted frame (9) has the integrability data

$$f_1 = \kappa_2 = 0,\ f_2 = -\bar{f}\cos^2\alpha,\ \kappa_1 = -\bar{f}\sin^2\alpha,\ \sigma = -\bar{f}\sin\alpha\cos\alpha \tag{47}$$

satisfying $f_2\kappa_1\sigma \neq 0,\ K^N = e_1(f_2) - f_2^2 \neq 0,\ \sigma = -e_1(\alpha),\ e_i(f_2) = e_i(\kappa_1) = e_i(\sigma) = e_i(K^{M^2}) = e_i(\bar{f}) = e_i(\alpha) = 0,\ i = 2, 3,$ and κ_1 solving the following PDE

$$\Delta\kappa_1 - \kappa_1\{-K^N + f_2^2\} = 0, \tag{48}$$

which is equivalent to

$$\alpha'''\sin\alpha\cos^2\alpha + \cos\alpha(\sin^2\alpha + 3)\alpha'\alpha'' + \sin\alpha(2\cos^2\alpha + 3)\alpha'^3 = 0, \tag{49}$$

where α', α'' and α''' denote the first, the second and the third derivative of α along the vector field e_1.

Proof. By Theorem 2.6, the local orthonormal frame given by (9) is adapted to the Riemannian submersion π with e_3 being vertical and the integrability data given by

$$\begin{cases} f_1 = 0,\ f_2 = -\bar{f}\cos^2\alpha,\ f_3 = -E_3(\theta) = 0,\ \sigma = -\bar{f}\sin\alpha\cos\alpha, \\ \kappa_1 = -\bar{f}\sin^2\alpha,\ \kappa_2 = \sin\alpha e_3(\cos\alpha) - \cos\alpha e_3(\sin\alpha) = -e_3(\alpha). \end{cases} \tag{50}$$

We may assume that $\cos\alpha = a_3^3 \neq \pm 1$, for otherwise $e_3 = \pm E_3$, π is the projection $M^2 \times \mathbb{R} \to M^2$ followed by a Riemannian covering map and hence π is harmonic. The rest of the proof will be done by considering the following three cases.

Case I: $a_3^3 = \cos\alpha = 0$. In this case, it follows from (50) that $f_1 = f_2 = \kappa_2 = \sigma = 0$, $\kappa_1 = -\bar{f} \neq 0$. From these and (6), we conclude that N^2 has Gauss curvature $K^N = 0$ and hence it is a flat surface. Therefore, biharmonic equation (8) turns into

$$\Delta\kappa_1 = 0.$$

It is easy to see that the above equation is equivalent to $\Delta^{M^2}\bar{f} = 0$ since $E_3\theta = 0$ and hence $E_3\bar{f} = 0$ and $\Delta\bar{f} = \Delta^{M^2}\bar{f} + E_3E_3\bar{f} = \Delta^{M^2}\bar{f}$.

Case II: $a_3^3 = \cos\alpha \neq 0, \pm 1$ and $f_2 = 0$. In this case, the second equation of (50) implies that $\bar{f} = 0$, and hence we have $f_1 = f_2 = f_3 = \kappa_1 = \sigma = K^{M^2} = K^N = 0$. Thus, we apply Corollary 2.7 to conclude that the biharmonic Riemannian submersion is harmonic in this case.

Case III: $a_3^3 = \cos\alpha \neq 0, \pm 1$ and $f_2 \neq 0$. In this case, the hypotheses can be summarized as:

$$a_2^3,\ a_3^3 \neq 0, \pm 1,\ f_2 \neq 0,\ a_1^3 = f_1 = f_3 = 0,\ e_3(f_1) = e_3(f_2) = 0. \tag{51}$$

Claim 1: In this case, we have $\kappa_2 = 0$, $e_2(\bar{f}) = e_3(\bar{f}) = e_2(f_2) = e_3(f_2) = e_2(\kappa_1) = e_3(\kappa_1) = e_2(\sigma) = e_3(\sigma) = e_2(K^{M^2}) = e_3(K^{M^2}) = e_2(\alpha) = e_3(\alpha) = 0$, $\kappa_1\sigma \neq 0$, and $\alpha' = -\sigma = \bar{f}\sin\alpha\cos\alpha$, where α' denotes the first derivative of α along $e_1 = \bar{E}_2$.

Proof of Claim 1: Firstly, we use (51), the 7th and the 8th equation of (24) to have $\kappa_1\sigma \neq 0$.

Secondly, by using (50) and the last equation of (51), we have $e_3(-\cos^2\alpha\bar{f}) = e_3(f_2) = 0$ and hence

$$e_3(\kappa_1) = e_3(-\sin^2\alpha\bar{f}) = e_3[-(1-\cos^2\alpha)\bar{f}] = -e_3(\bar{f}). \tag{52}$$

We use (50), the 5th and the 7th equation of (5) with $a_1^3 = 0$ to obtain

$$\sigma^2 = \kappa_1 f_2,\ e_2(\kappa_2) = \kappa_2^2,\ e_2(\sigma) = 2\kappa_2\sigma. \tag{53}$$

On the other hand, note that $a_2^3 = \sin\alpha,\ a_3^3 = \cos\alpha$, a straightforward computation using the 6th equation of (24) gives

$$0 = e_3(f_2) = e_3(-\cos^2\alpha \bar{f}) = -2\kappa_2 \sin\alpha\cos\alpha \bar{f} - \cos^2\alpha e_3(\bar{f}).$$

It follows that

$$e_3(\bar{f}) = \tfrac{-2\kappa_2 \sin\alpha\cos\alpha \bar{f}}{\cos^2\alpha} = \tfrac{2\kappa_2\sigma}{\cos^2\alpha}. \tag{54}$$

Using (52), (54), the 1st equation of (53) and the 3rd equation of (5) with $a_1^3 = f_1 = 0$, we obtain

$$\begin{aligned} &e_3(\sigma) = \tfrac{e_3(\sigma^2)}{2\sigma} = \tfrac{f_2 e_3(\kappa_1)}{2\sigma} = \kappa_2 \bar{f},\\ &e_1(\kappa_2) = e_3(\sigma) + \kappa_1\kappa_2 = \kappa_2\bar{f} - \kappa_2\sin^2\alpha\bar{f} = \kappa_2\cos^2\alpha\bar{f} = -\kappa_2 f_2. \end{aligned} \tag{55}$$

On the other hand, using $e_2(\cos\alpha) = e_2(\sin\alpha) = 0$, $\bar{f} = -\frac{\sigma}{\sin\alpha\cos\alpha}$ and $e_2(\sigma) = 2\kappa_2\sigma$ and a simple computation we have

$$\begin{aligned} &e_2(\bar{f}) = -\tfrac{e_2(\sigma)}{\sin\alpha\cos\alpha} = \tfrac{-2\kappa_2\sigma}{\sin\alpha\cos\alpha} = 2\kappa_2\bar{f},\\ &e_2(f_2) = e_2(-\cos^2\alpha\bar{f}) = -\cos^2\alpha e_2(\bar{f}) = -2\kappa_2\cos^2\alpha\bar{f} = 2\kappa_2 f_2,\\ &e_2(\kappa_1) = e_2(-\sin^2\alpha\bar{f}) = -\sin^2\alpha e_2(\bar{f}) = -2\kappa_2\sin^2\alpha\bar{f} = 2\kappa_1\kappa_2. \end{aligned} \tag{56}$$

Note that $\kappa_2 e_3(f_2) = [e_2, e_3](f_2) = e_2 e_3(f_2) - e_3 e_2(f_2)$, a further computation using (56) gives $0 = -e_3(2\kappa_2 f_2) = -f_2 e_3(\kappa_2)$, which, together with $f_2 \neq 0$, implies that

$$e_3(\kappa_2) = 0. \tag{57}$$

Applying the 2nd equation of biharmonic equations (8) and using (4), (53), (55), (56), (57) and (6) with $f_1 = f_3 = 0$, we have

$$\begin{aligned} 0 &= -\Delta\kappa_2 + 2f_2 e_2(\kappa_1) + \kappa_1 e_2(f_2) - \kappa_1\kappa_2 f_2 + \kappa_2\{-K^N + f_2^2\}\\ &= \sum_{i=1}^{3}\{-e_i e_i(\kappa_2) + \nabla_{e_i} e_i(\kappa_2)\} + 4\kappa_1\kappa_2 f_2 + 2\kappa_1\kappa_2 f_2 - \kappa_1\kappa_2 f_2 + \kappa_2\{-e_1(f_2) + 2f_2^2\}\\ &= -\kappa_2^3 + 4\kappa_1\kappa_2 f_2 = \kappa_2(-\kappa_2^2 + 4\kappa_1 f_2) = \kappa_2(-\kappa_2^2 + 4\sigma^2). \end{aligned} \tag{58}$$

Solving (58) we must have $\kappa_2 = 0$ and $\kappa_2^2 \neq 4\sigma^2$. Indeed, if $\kappa_2^2 = 4\sigma^2$, then using (57), we have $4e_3(\sigma^2) = e_3(\kappa_2^2) = 0$ and hence $e_3(\sigma) = 0$. But since $\bar{f} \neq 0$ and (55), we further have $\kappa_2 = 0$ and hence $4\sigma^2 = \kappa_2^2 = 0$, a contradiction, since $\sigma \neq 0$. Therefore, a straightforward computation using (52)–(56), the 5th, the 6th equation equation of (24) gives

$$e_2(\bar{f}) = e_3(\bar{f}) = e_3(\sin\alpha) = e_3(\cos\alpha) = e_2(\kappa_1) = e_3(\kappa_1) = e_3(\sigma) = e_2(\sigma) = e_2(f_2) = 0. \tag{59}$$

Since $K^{M^2} = -\bar{E}_2(\bar{f}) - \bar{f}^2 = -e_1(\bar{f}) - \bar{f}^2$, we see that $e_3(K^{M^2}) = e_2(K^{M^2}) = 0$. From (24) and (59), we have $e_2(\alpha) = e_3(\alpha) = 0$, and a direct computation gives

$\sigma \sin\alpha = e_1(\cos\alpha) = -\sin\alpha\alpha'$, i.e., $\alpha' = -\sigma = \sin\alpha\cos\alpha\bar{f}$, where α' denotes the first derivative of α along $e_1 = \bar{E}_2$. For $\kappa_2 = f_1 = 0$, biharmonic equation (8) turns into (48). A further computation using Claim 1 and (50), we see that (48) turns into (49).

Finally, we show that (N^2, h) is not flat. If otherwise, $K^N = 0$, i.e., $e_1(f_2) - f_2^2 = 0$, using the 4th equation of (5) with $\sigma = -\bar{f}\sin\alpha\cos\alpha$, we immediately have

$$K^{M^2} = -3\bar{f}^2\sin^2\alpha. \tag{60}$$

On the other hand, we know that M^2 has Gauss curvature

$$K^{M^2} = -\bar{E}_2(\bar{f}) - \bar{f}^2 = -e_1(\bar{f}) - \bar{f}^2. \tag{61}$$

Using Equations (60) and (61), we obtain

$$\bar{f}' = 3\bar{f}^2\sin^2\alpha - \bar{f}^2, \tag{62}$$

where $\bar{f}'$ denotes the first derivative of $\bar{f}$ along e_1.
Substituting $\alpha' = \sin\alpha\cos\alpha\bar{f}$ (i.e., $\bar{f} = \frac{\alpha'}{\sin\alpha\cos\alpha}$) into (62) and simplifying the resulting equation we get

$$\alpha''\cos\alpha - \alpha'^2\sin\alpha = 0. \tag{63}$$

One applies e_1 to both sides of (63) and simplifies the resulting equation to have

$$\alpha'''\cos\alpha - 3\alpha'\alpha''\sin\alpha - \alpha'^3\cos\alpha = 0. \tag{64}$$

Adding (49) to a $(-\sin\alpha\cos\alpha)$ multiple of (64) and simplifying the results with $\alpha' \neq 0$ yields

$$\alpha''\cos\alpha(4\sin^2\alpha + 3) + \alpha'^2\sin\alpha(3\cos^2\alpha + 3) = 0. \tag{65}$$

Similarly, adding a $(-4\sin^2\alpha - 3)$ multiple of (63) to (66) and simplifying the results gives

$$\alpha'^2\sin\alpha(9 + \sin^2\alpha) = 0, \tag{66}$$

it follows that α is a constant, a contradiction. Then, we must have $K^N \neq 0$.

Summarizing all results in the above cases we obtain the theorem. □

We now give a characterization of proper biharmonic Riemannian submersions from $M^2 \times \mathbb{R}$ by using the local coordinates as follows

Theorem 2.9. *If $\pi : M^2 \times \mathbb{R} \to (N^2, h)$ is a proper biharmonic Riemannian submersion from the product space, then* *(i) The target surface is flat, and locally, up to an isometry of the domain and/or codomain, π is the projection of the special twisted product*

$$\pi : (\mathbb{R}^3, e^{2p(x,y)}dx^2 + dy^2 + dz^2) \to (\mathbb{R}^2, dy^2 + dz^2),\ \pi(x, y, z) = (y, z), \tag{67}$$

with $p_y \neq 0$ being a harmonic function on $(M^2, e^{2p(x,y)}dx^2 + dy^2)$, i.e., it solves the PDE

$$\Delta p_y := p_{yyy} + p_{yy}p_y + e^{-2p(x,y)}(p_{xxy} - p_{xy}p_x) = 0. \tag{68}$$

or,
(ii) The target surface is non-flat, and locally, up to an isometry of the domain and/or codomain, the map can be expressed as

$$\begin{aligned} &\pi : (\mathbb{R}^3, e^{2p(x,y)}dx^2 + dy^2 + dz^2) \to (\mathbb{R}^2, dy^2 + e^{2\lambda(y,\phi)}d\phi^2), \\ &\pi(x, y, z) = (y, F\left(z - \textstyle\int e^{\varphi(x)}dx\right)), \end{aligned} \tag{69}$$

where $p(x, y) = \ln|\tan\alpha(y)| + \varphi(x)$, $\lambda = \ln|\sin\alpha(y)| + w(\phi)$ with the functions $\varphi(x)$, $w(\phi)$ and nonconstant function $F(u)$ satisfying $F'(z - \int e^{\varphi(x)}dx) = e^{-w(\phi)}$ and $z - \int e^{\varphi(x)}dx = \int e^{w(\phi)}d\phi$, and $\alpha(y)$ is the angle between the fibers of π and $E_3 = \partial_z$ solving the ODE (49).

Proof. First of all, note that by Theorem 2.8 the local orthonormal frame $\{e_1 = \bar{E}_2,\ e_2 = -\cos\alpha\bar{E}_1 + \sin\alpha\bar{E}_3,\ e_3 = \sin\alpha\bar{E}_1 + \cos\alpha\bar{E}_3\}$ is an adapted frame of the Riemannian submersion π, and that the vector field $e_1 = \bar{E}_2$ is a geodesic vector field on M^2. It is well known that we can choose local semi geodesic coordinates (x, y) on M^2 so that $\bar{E}_1 = e^{-p(x,y)}\partial_x, \bar{E}_2 = \partial_y$, and the metric on M^2 takes the form $e^{2p(x,y)}dx^2 + dy^2$. It follows that the product manifold $M^2 \times \mathbb{R}$ can be locally represented as $(U \times \mathbb{R} \subseteq M^2 \times \mathbb{R}, e^{2p(x,y)}dx^2 + dy^2 + dz^2)$.

For Statement (i), since the target surface is flat, it corresponds to the Case I in the proof of Theorem 2.8, i.e., the frame $\{e_1 = \bar{E}_2,\ e_2 = E_3,\ e_3 = \bar{E}_1\}$ is an adapted frame to the Riemannian submersion π with the integrability data $\{f_1 = f_2 = \kappa_2 = \sigma = 0,\ \kappa_1 = -\bar{f} \neq 0\}$. Note that, by $\sigma = 0$ and the 4th equation of (5), the horizontal distribution of the Riemannian submersion is integrable with flat integral submanifolds. So locally, up to an isometry of the domain and/or the target manifold, the Riemannian submersion is the projection along the fibers (i.e., the integral curves of $\bar{E}_1 = e^{-p(x,y)}\partial_x$ to the integral submanifold, and hence can be described by (67)). It is easily checked that in this case $\kappa_1 = -\bar{f} = -p_y$ and Equation (46) reduces to (68).

For Statement (ii), Theorem 2.8 implies that in this case, the target surface is non-flat, $\cos\alpha \neq$ constant depending only on variable y, and $\{e_1 = \bar{E}_2,\ e_2 =$

$-\cos\alpha\bar{E}_1+\sin\alpha\bar{E}_3,\ e_3=\sin\alpha\bar{E}_1+\cos\alpha\bar{E}_3\}$ is an adapted frame of the Riemannian submersion π with integrability data (47).

Note that the vector field $e_1=\bar{E}_2=\partial_y$ is a basic vector field to the Riemannian submersion π. It is well known that there is a local vector field ε_1 on (N^2,h) whose integral curves are geodesics on (N,h) such that $d\pi(e_1)=\varepsilon_1$. It follows that we can choose a local semi geodesic coordinates (y,ϕ) on N so that metric takes the form $h=dy^2+e^{2\lambda(y,\phi)}d\phi^2$, and the orthonormal frame $\varepsilon_1=\partial_y, \varepsilon_2=e^{-\lambda}\partial_\phi$ satisfying $d\pi(e_1)=\varepsilon_1, d\pi(e_2)=\varepsilon_2=e^{-\lambda}\partial_\phi$.

Summarizing the above, we conclude that in this case, up to and isometry of the domain and/or target manifold, the Riemannian submersion can be expressed as

(70)
$$\pi:(\mathbb{R}^3,e^{2p(x,y)}dx^2+dy^2+dz^2)\to(\mathbb{R}^2,h=dy^2+e^{2\lambda(y,\phi)}d\phi^2),\ \pi(x,y,z)=(y,\phi),$$

where $\phi=\phi(x,y,z)$ is a function to be determined.

Now we are to determine the functions $\phi=\phi(x,y,z)$, $p(x,y)$, and $\lambda(y,\phi)$.

A straightforward computation gives

$$F_1\circ\pi\varepsilon_1+F_2\circ\pi\varepsilon_2=[\varepsilon_1,\varepsilon_2]=-\lambda_y\varepsilon_2, \tag{71}$$

which implies

$$F_1=0,\ F_2=-\lambda_y.$$

Hence, we have

$$\lambda_y=-F_2\circ\pi=-f_2=\bar{f}\cos^2\alpha=\tfrac{\cos\alpha\alpha'(y)}{\sin\alpha}. \tag{72}$$

Integrating both sides with respect to y yields

$$\lambda(y,\phi)=\int\tfrac{\cos\alpha\alpha'(y)}{\sin\alpha}dy+w(\phi)=\ln|\sin\alpha(y)|+w(\phi), \tag{73}$$

where $w=w(\phi)$ is an arbitrary function on ϕ.

Noting that $p_y=\bar{f}(y)=\frac{\alpha'(y)}{\sin\alpha\cos\alpha}$ does not depend on x we have

$$p(x,y)=\int\bar{f}(y)dy+\varphi(x)=\int\tfrac{\alpha'(y)}{\sin\alpha\cos\alpha}dy+\varphi(x)=\ln|\tan\alpha(y)|+\varphi(x), \tag{74}$$

where $\varphi=\varphi(x)$ is a function on x.

To determine the component function $\phi(x,y,z)$, we use $e_1=\partial_y, e_2=-\cos\alpha e^{-p(x,y)}\partial_x+\sin\alpha\partial_z, e_3=\sin\alpha e^{-p(x,y)}\partial_x+\cos\alpha\partial_z$, and $d\pi(e_1)=\varepsilon_1=\partial_y$, $d\pi(e_2)=\varepsilon_2=e^{-\lambda}\partial_\phi$, and $d\pi(e_3)=0$ to have

$$\begin{aligned}\partial_y=\varepsilon_1 &= d\pi(e_1)=\partial_y+\phi_y\,\partial_\phi,\\ e^{-\lambda}\tfrac{\partial}{\partial\phi}=\varepsilon_2 &= d\pi(e_2)=(-\cos\alpha e^{-p(x,y)}\phi_x+\sin\alpha\phi_z)\partial_\phi,\\ 0= &\ d\pi(e_3)=(\sin\alpha e^{-p(x,y)}\phi_x+\cos\alpha\phi_z)\partial_\phi.\end{aligned} \tag{75}$$

By comparing both sides of the 1st equation of (75), one finds that $\phi_y = \frac{\partial \phi}{\partial y} = 0$, which means that the function ϕ does not depend on y, i.e., $\phi = \phi(x, z)$.
Comparing coefficients of both sides of the 2nd and the 3rd equation of (75) separately, we get

$$-\cos\alpha e^{-p(x,y)}\phi_x + \sin\alpha\phi_z = e^{-\lambda},\ \sin\alpha e^{-p(x,y)}\phi_x + \cos\alpha\phi_z = 0. \tag{76}$$

Recall that $p(x, y) = \ln|\tan\alpha(y)| + \varphi(x)$, $\lambda = \ln|\sin\alpha(y)| + w(\phi)$, and the fact that $\phi(x, z)$ is nonconstant since $d\pi(e_2) \neq 0$, we use the method of the first integral to solve the 2nd PDE of (76) to have

$$\phi(x, z) = F\left(z - \int e^{\varphi(x)}dx\right), \tag{77}$$

where $F = F(u)$ is a nonconstant differentiable function. Substituting this into the 1st PDE of (76) we have $F'(z - \int e^{\varphi(x)}dx) = e^{-w(\phi)}$. It follows from this and (77) that $d\phi = dF = F'du = e^{-w(\phi)}du$ and hence $e^{w(\phi)}d\phi = du$ implying that $u = z - \int e^{\varphi(x)}dx = \int e^{w(\phi)}d\phi$. This completes the proof of Statement (ii). □

Applying Theorem 2.9, we immediately have the following corollary which characterizes a proper biharmonic Riemannian submersions from a product manifold onto a non-flat surface as a special map determined up to an arbitrary function between two special warped product manifolds with the warping functions solving an ODE.

Corollary 2.10. *A proper biharmonic Riemannian submersion $\pi : M^2 \times \mathbb{R} \to (N^2, h)$ from product manifold into a non-flat surface is locally, up to an isometry of the domain and/or codomain, π is a map between two special warped product spaces given by*

$$\begin{aligned}&\pi : (\mathbb{R}^3, \tan^2\alpha(y)\,dt^2 + dy^2 + dz^2) \to (\mathbb{R}^2, dy^2 + \sin^2\alpha(y)\,d\psi^2)\\ &\pi(t, y, z) = (y, z - t),\end{aligned} \tag{78}$$

where $\alpha(y)$ is the angle between the fibers of π and $E_3 = \frac{\partial}{\partial z}$ solving the ODE

$$\alpha''' \sin\alpha\cos^2\alpha + \cos\alpha(\sin^2\alpha + 3)\alpha'\alpha'' + \sin\alpha(2\cos^2\alpha + 3)\alpha'^3 = 0, \tag{79}$$

Proof. This follows from Statement (ii) of Theorem 2.9 and the coordinate changes $t = \int e^{\varphi(x)}dx,\ y = y,\ z = z$ in the domain and $y = y, \psi = \int e^{w(\phi)}d\phi$ in the codomain. □

Remark 3. (A) Note that it follows form [1] (Corollary 3.2) that the Riemannian submersion given by the projection of the twisted product $\pi : (\mathbb{R}^3, e^{2p(x,y,z)}dx^2 + dy^2 + dz^2) \to (\mathbb{R}^2, dy^2 + dz^2)$, $\pi(x, y, z) = (y, z)$ is biharmonic if and only if $\Delta\kappa_1 = 0, \Delta\kappa_2 = 0$. In the case of (i) in Theorem 2.9, these reduce exactly to (68).

Thus, Statement (i) of Theorem 2.9 not just characterizes proper biharmonic Riemannian submersions from product manifold to a flat surface locally but also recovers a special result in [1] (Corollary 3.2).
(B) The biharmonicity of the Riemannian submersion defined by the projection of warped product $\pi : (\mathbb{R}^3, e^{2p(y,z)}dx^2 + dy^2 + dz^2) \to (\mathbb{R}^2, dy^2 + dz^2),\ \pi(x, y, z) = (y, z)$ had been studied in [2, 15, 21, 12, 1].

(D) We would like to point out that, unlike in case (ii) in Theorem 2.9 and Corollary 2.10 where the function $p(x, y)$ can be proved to be independent of x variable so the metric is of a warped product type, in case (i) the map is completely determined (to be the projection) but we do not know whether the function $p(x, y)$ is independent of x variable or not.

When the target surface is flat, it is easy to have many examples proper biharmonic Riemannian submersions $\pi : M^2 \times \mathbb{R} \to \mathbb{R}^2$ from the projection of the warped product spaces see e.g., [2, 15, 12, 1]. For example, the following projections are proper biharmonic Riemannian submersions:

(*i*) $\pi : (\mathbb{R}^2 \times \mathbb{R}, (\cosh y)^4dx^2 + dy^2 + dz^2) \to (\mathbb{R}^2, dy^2 + dz^2),\ \pi(x, y, z) = (y, z)$,

(*ii*) $\pi : (\mathbb{R}^2_+ \times \mathbb{R}, y^4dx^2 + dy^2 + dz^2) \to (\mathbb{R}^2_+, dy^2 + dz^2),\ \pi(x, y, z) = (y, z)$.

When the target surface is non-flat, we would like to point out that there exist many local proper biharmonic Riemannian submersions $\pi : M^2 \times \mathbb{R} \to (N^2, h)$. In fact, by introducing new variable $u(\alpha) = \frac{\alpha''(y)}{\alpha'(y)^2}$, we have $\alpha'\alpha'' = u\,\alpha'^3$, $\alpha''' = (u' + 2u^2)a'^3$ and hence (79) reduces to a Riccati equation as

$$u'(\alpha) + 2u^2 + \tfrac{\sin^2\alpha+3}{\sin\alpha\cos\alpha}u + \tfrac{2\cos^2\alpha+3}{\cos^2\alpha} = 0, \tag{80}$$

any solution of which gives a family of locally defined proper biharmonic Riemannian submersions $M^2 \times \mathbb{R} \to (N^2, h)$.

Finally, note that it was proved in [21, 22] that a proper biharmonic Riemannian submersion $M^2(c)\times\mathbb{R}$ exists only in the case when $c < 0$, and $\pi : (\mathbb{R}^3, e^{2\sqrt{-c}\,y}dx^2 + dy^2 + dz^2) \to (\mathbb{R}^2, dy^2 + dz^2),\ \pi(x, y, z) = (y, z)$ is an example. Now we can prove that, up to isometry, this is the only one.

Proposition 2.11. *A Riemannian submersion* $\pi : M^2(c)\times\mathbb{R} \to (N^2, h)$ *is proper biharmonic if and only if* $c < 0$, (N^2, h) *is flat, and, up to an isometry, the map can be expressed as* $\pi : H^2(c) \times \mathbb{R} \to \mathbb{R}^2$ *with* $\pi : (\mathbb{R}^3, e^{2\sqrt{-c}\,y}dx^2 + dy^2 + dz^2) \to (\mathbb{R}^2, dy^2 + dz^2),\ \pi(x, y, z) = (y, z)$.

Proof. Firstly, it follows from [22] that a proper biharmonic Riemannian submersion $\pi : M^2(c) \times \mathbb{R} \to (N^2, h)$ from a product space exists only in the case:

$H^2(c) \times \mathbb{R} \to \mathbb{R}^2$ with $c < 0$.

Secondly, by Theorem 2.8 and 2.9, we know that locally, up to an isometry of the domain and/or codomain, a proper biharmonic Riemannian submersion $\pi : H^2(c) \times \mathbb{R} \to \mathbb{R}^2$ with $c < 0$ is expressed as

$$\pi : (\mathbb{R}^3, e^{2p(x,y)}dx^2 + dy^2 + dz^2) \to (\mathbb{R}^2, dy^2 + dz^2),\ \pi(x, y, z) = (y, z), \tag{81}$$

and the orthonormal frame $\{e_1 = \partial_y,\ e_2 = \partial_z,\ e_3 = e^{-p}\partial_x\}$ is adapted to the Riemannian submersion π with the integrability data $f_1 = f_2 = \kappa_2 = \sigma = 0,\ \kappa_1 = -p_y \neq 0$. It is easily checked that in this case, (5) reduces to

$$e_1(\kappa_1) = \kappa_1^2 + c,\ e_2(\kappa_1) = 0, \tag{82}$$

and biharmonic equation (8) reads

$$\Delta\kappa_1 = 0. \tag{83}$$

A straightforward computation gives

$$\begin{aligned}\Delta\kappa_1 &= e_1e_1(\kappa_1) + e_3e_3(\kappa_1) - \nabla_{e_1}e_1(\kappa_1) - \nabla_{e_2}e_2(\kappa_1) - \nabla_{e_3}e_3(\kappa_1)\\ &= e_1(\kappa_1^2 + c) + e_3e_3(\kappa_1) - \kappa_1e_1(\kappa_1) = e_3e_3(\kappa_1) + \kappa_1^3 + c\kappa_1.\end{aligned} \tag{84}$$

Substituting (84) into (83), we have

$$e_3e_3(\kappa_1) = -\kappa_1^3 - c\kappa_1. \tag{85}$$

Applying e_3 to both sides of the 1st equation of (82) and using the fact that $e_1e_3(\kappa_1) = [e_1, e_3](\kappa_1) + e_3e_1(\kappa_1)$, we have

$$e_1e_3(\kappa_1) = 3\kappa_1e_3(\kappa_1). \tag{86}$$

Using (82), (85), (86), and a direct computation we get

$$e_1e_3\{e_3(\kappa_1)\} - e_3e_1\{e_3(\kappa_1)\} = [e_1, e_3]\{e_3(\kappa_1)\} = \kappa_1e_3e_3(\kappa_1) = -\kappa_1^4 - c\kappa_1^2, \tag{87}$$

and

$$\begin{aligned}&e_1e_3\{e_3(\kappa_1)\} - e_3e_1\{e_3(\kappa_1)\} = e_1\{e_3e_3(\kappa_1)\} - e_3\{e_1e_3(\kappa_1)\}\\ &= -c\kappa_1^2 - 4c - 3e_3^2(\kappa_1).\end{aligned} \tag{88}$$

Comparing (87) with (88), we get

$$3e_3^2(\kappa_1) = \kappa_1^4 - 4c. \tag{89}$$

Applying e_3 to both sides of (89) and using (85) to simplify the resulting equation we have

$$\kappa_1(5\kappa_1^2 + 3c)e_3(\kappa_1) = 0, \tag{90}$$

which implies $e_3(\kappa_1) = 0$. Substituting this into (85) and using that fact that $c\kappa_1 \neq 0$ we obtain

$$\kappa_1^2 = -c > 0, \tag{91}$$

which implies that

$$p_y^2 = -c, \text{ (and hence) } p_y = \pm\sqrt{-c}. \tag{92}$$

It follows that

$$p(x,y) = \pm\sqrt{-c}y + \varphi(x), \tag{93}$$

where $\varphi(x)$ is a an arbitrary function.

So we conclude that up to an isometry of the domain and/or codomain, a proper biharmonic Riemannian submersion $\pi : H^2(c) \times \mathbb{R} \to \mathbb{R}^2$ with $c < 0$ is expressed as

$$\pi : (\mathbb{R}^3, e^{2\sqrt{-c}\,y}dx^2 + dy^2 + dz^2) \to (\mathbb{R}^2, dy^2 + dz^2),\ \pi(x,y,z) = (y,z). \tag{94}$$

Thus, we obtain the proposition. □

Remark 4. By Proposition 2.11, for $c \geq 0$, there exists no proper biharmonic Riemannian submersion $\pi : M^2(c) \times \mathbb{R} \to (N^2, h)$ no matter what (N^2, h) is.

References

[1] M. A. Akyol and Y. -L. Ou, *Biharmonic Riemannian submersions*, Annali di Matematica Pura ed Applicata, (2019) 198:559-570.

[2] A. Balmus, S. Montaldo and C. Oniciuc, *Biharmonic maps between warped product manifolds*, J. Geom. Phys. 57 (2007) 449-466.

[3] A. Balmus, S. Montaldo and C. Oniciuc, *Classification results for biharmonic submanifolds in spheres*, Israel J. Math. 168 (2008), 201–220.

[4] P. Baird and J. C. Wood, *Harmonic morphisms between Riemannian manifolds*, London Math. Soc. Monogr. (N.S.) No. 29, Oxford Univ. Press (2003).

[5] P. Baird and J. C. Wood, *Bernstein theorems for harmonic morphisms from R^3 and S^3*, Math. Ann. 280 (1988), no. 4, 579–603.

[6] R. Caddeo, S. Montaldo, and C. Oniciuc, *Biharmonic submanifolds of S^3*. Internat. J. Math. 12 (2001), no. 8, 867–876.

[7] R. Caddeo, S. Montaldo and C. Oniciuc, *Biharmonic submanifolds in spheres*, Israel J. Math. 130 (2002), 109–123.

[8] B.Y. Chen, *Some open problems and conjectures on submanifolds of finite type*, Soochow J. Math. 17 (1991), 169-188.

[9] J. Eells and L. Lemaire, Selected Topics in harmonic maps, CBMS, Regional Conference Series in Math., Amer. Math. Soc., 50, 1983.

[10] D. Fetcu, C. Oniciuc and H. Rosenberg, *Biharmonic submanifolds with parallel mean curvature in $\mathbb{S}^n \times \mathbb{R}$*, J. Geom. Anal. (2013) 23, 2158–2176.

[11] Y. Fu, S. Maeta, and Y.-L. Ou, *Biharmonic hypersurfaces in a product space* $L^m \times \mathbb{R}$, Math Nach. 294(9), 1724-1741.

[12] E. Ghandour and Y.-L. Ou, Generalized harmonic morphisms and horizontally weakly conformal biharmonic maps, J. Math. Anal. Appl. 464(1), 924–938 (2018).

[13] G. Y. Jiang, 2-*Harmonic maps and their first and second variational formulas*, Chin. Ann. Math. Ser. A 7(1986) 389-402.

[14] G. Y. Jiang, *Some non-existence theorems of* 2-*harmonic isometric immersions into Euclidean spaces* , Chin. Ann. Math. Ser. 8A (1987) 376-383.

[15] E. Loubeau and Y.-L. Ou, *Biharmonic maps and morphisms from conformal mappings*, Tôhoku Math. J. 62, 55-73 (2010).

[16] C. Oniciuc, *Biharmonic maps between Riemannian manifolds*, An. Stiint. Univ. Al. I. Cuza Iasi. Mat. (N.S.) 48(2), 237-248 (2002)

[17] Y.-L. Ou and Z.-P. Wang, *Constant mean curvature and totally umbilical biharmonic surfaces in 3-dimensional geometries*, J. Geom. Phys. 61(2011), 1845-1853.

[18] Y.-L. Ou and Tang. L, *On the generalized Chen's conjecture on biharmonic submanifolds*, Michigan Math. J., 61 (2012), 531-542.

[19] Y.-L. Ou and B.-Y. Chen, *Biharmonic submanifolds and biharmonic maps in Riemannian Geometry*, World Scientific Publishing Co. Pte. Ltd., Hackensack, N. J., 2020.

[20] H. Urakawa, *Harmonic maps and biharmonic maps on principal bundles and warped products*, J. Korean Math. Soc. 55 (2018), No. 3, pp. 553-574.

[21] Z.-P. Wang and Y.-L. Ou, *Biharmonic Riemannian submersions from 3-manifolds*, Math. Z. (2011) 269:917-925.

[22] Z.-P. Wang and Y.-L. Ou, *Biharmonic Riemannian submersions from a 3-dimensional BCV space*, J. of Geom. Anal., 2024, to appear.

[23] Z.-P. Wang, Y.-L. Ou and Y.-G. Luo, *Harmonic Riemannian submersions from 3-dimensional geometries*, preprint, 2023, https://doi.org/10.48550/arXiv.2302.09600.

School of Mathematical Sciences,
Guizhou Normal University,
Guiyang 550025,
People's Republic of China
E-mail:zpwzpw2012@126.com (Wang),

Department of Mathematics,
Texas A & M University-Commerce,
Commerce TX 75429,
USA.
E-mail:yelin_ou@tamu-commerce.edu (Ou).