The uniqueness of Leray-Hopf solution for a Class of incompressible Navier-Stokes Equations in three spatial dimensions

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Abstract

In the case of external force $f \in C_0^\infty([0,\infty); C_0^\infty(\Omega)^3)$ and the initial data $u^0 \in W_{0,\sigma}^{1,2}(\Omega)$ without the Serrin's condition, we show the uniqueness of *Leray-Hopf solution* to incompressible Navier-Stokes Equations in Theorem 3.1.

Keywords. Navier-Stokes equation, weak solution, Leray-Hopf solutions, Serrin's condition.
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1 Introduction

This paper is concerned with the uniqueness of Leray-Hopf solutions to incompressible Navier-Stokes Equations in three spatial dimensions with external force f:

$$(1.1)\qquad \begin{cases} u_t + u\cdot\nabla u = \nu\Delta u - \nabla p + f, & t>0, x\in\Omega,\\ \text{div } u = \displaystyle\sum_{i=1}^{3}\frac{\partial u_i}{\partial x_i} = 0, & t\ge 0, x\in\Omega,\\ u(0,x) = u^0(x), & x\in\Omega,\\ u(t,x) = 0, & t\ge 0, x\in\partial\Omega. \end{cases}$$

where $\Omega = \mathbb{R}^3$ or $\Omega(\subset \mathbb{R}^3)$ be an open nonempty connected bounded uniform C^2-domain and C^∞-domain. The boundary condition $u|_{\partial\Omega} = 0$ is omitted if $\Omega = \mathbb{R}^3$, the initial data $u^0 = (u_1^0, u_2^0, u_3^0) \in W_{0,\sigma}^{1,2}(\Omega)$, the external force

$$f = (f_1, f_2, f_3) \in C_0^\infty([0,\infty); C_0^\infty(\Omega)^3) := \{u|_{[0,\infty)\times\Omega}; u \in C_0^\infty((-1,\infty)\times\Omega)^n\},$$

and ν is a positive constant called the *viscosity*. The unknown velocity function $u = u(t,x) = (u_1(t,x), u_2(t,x), u_3(t,x))$ means a three-component vector field and the unknown pressure $p = p(t,x)$ be a scalar field. The divergence free condition on u represents the incompressibility of the fluid. In the above, the convection operator and the divergence operator are defined as follows

$$u\cdot\nabla u := (u\cdot\nabla)u = \sum_{i=1}^{3} u_i\frac{\partial u}{\partial x_i}, \qquad \text{div } u := \nabla\cdot u := \sum_{i=1}^{3}\frac{\partial u_i}{\partial x_i}.$$

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It is well known that, for $u^0 \in L^2_\sigma(\Omega)$, $f \in L^1(0,T;L^2(\Omega)^3)$, the problem (1.1) possesses at least one global weak solution u satisfying the energy inequality $\frac{1}{2}||u(t)||_2^2 + \nu \int_0^t ||\nabla u||_2^2 d\tau \le \frac{1}{2}||u^0||_2^2 + \int_0^t \langle f, u\rangle \, d\tau$, for any $t \ge 0$(see Hopf([5]), J.Leray([9]), Sohr([14])). Such solutions are called *Leray-Hopf solutions*. The regularity and uniqueness of weak solutions are often interrelated. It turns out that if the space dimension $n = 2$, *Leray-Hopf solutions* are unique and regular (see Lions-Prodi([12]), Lions([11]), Ladyzhenskaya([7]), Serrin([18]), Temam([19])); However, for $n \ge 3$ the uniqueness and regularity of *Leray-Hopf solutions* becomes exceptionally difficult. On the one hand, there are many important results concerning the uniqueness and the regularity of Leray-Hopf solutions. Among them we wish to mention the work of Serrin([18]) which asserts that if a *Leray-Hopf solution* u of (1.1) also satisfies Serrin's condition, i.e., $u \in L^s_{loc}(0,T;(L^q(\Omega))^n)$ for some $2 < s < \infty, n < q < \infty$, so that $2/s + n/q < 1$, then, *Leray-Hopf solutions* of (1.1) are unique, after a redefinition on a null set of $[0,T) \times \Omega$, $u \in (C^\infty((0,T)\times\Omega))^n$, (namely, u be a strong solutions). This result was later improved by Fabes, E.B. and Riviere, N.M.([4]), Struwe,M.([13]) to the case of equality. On the other hand, in 1969, Ladyzhenskaya [8] already gave an example of non-uniqueness to (1.1), though in a time-varying domain which degenerates as $t \to 0^+$ and with non-standard boundary conditions and a force. In recent years, this problem has been revisited. We mention one of particular importance: Dallas Albritton., Elia Brué., Maria Colombo in [1] have proven the existence of two distinct suitable Leray–Hopf solutions u, $\bar{u}$ to the Navier–Stokes equations (1.1) on $\mathbb{R}^3 \times (0,T)$ with body force $f \in L^1(0,T;L^2(\mathbb{R}^3)^3)$ and initial condition $u^0 \equiv 0$. In this paper, we will prove that Leray-Hopf solutions of the equation (1.1) is unique in the case of external force $f \in C_0^\infty([0,\infty);C_0^\infty(\Omega)^3)$ and the initial data $u^0 \in W^{1,2}_{0,\sigma}(\Omega)$ without the Serrin's condition. It is worth pointing out that our conclusion is not contradictory with the paper of Dallas Albritton (see Theorem 1.3 of [1]).

The rest of this paper is organized as follows. We shall first give problem statement and preliminaries in Section 2. Then, in Section 3, we will show the uniqueness of *Leray-Hopf solutions* for a class of NSE (1.1) in Theorem 3.1. In Section 4, we show the existence of weak solutions of a particular Navier-Stokes Equations (3.11) and the energy inequality.

2 Preliminaries

In this section we will give the necessary denotations, definitions and some former conclusions. Throughout this section, $\Omega \subseteq \mathbb{R}^n$ means a general domain, that is any open nonempty connected subset of the n-dimensional Euclidean space $\mathbb{R}^n$, $(n = 3)$.

Given a non-negative integer k, let $C^k(\Omega)$ denote the linear space of all real functions u defined in Ω which together with all their derivatives $D^\alpha u$ of order $\alpha \le k$ are continuous in Ω. $C_0^k(\Omega) := \{u \in C^k(\Omega); supp u (\subset \Omega) \text{ compact}\}$, $C^\infty(\Omega) := \bigcap_{k\ge0} C^k(\Omega)$, $C_0^\infty(\Omega) := \bigcap_{k\ge0} C_0^k(\Omega)$. We consider the Hilbert space $L^2(\Omega)^n$ with scalar product $\langle f, g\rangle = \int_\Omega f \cdot g dx$, the subspace

$$L^2_\sigma(\Omega) := \overline{C^\infty_{0,\sigma}(\Omega)}^{||\cdot||_2}, \quad C^\infty_{0,\sigma}(\Omega) := \{f \in C_0^\infty(\Omega)^n; \mathrm{div} f = 0\},$$

and the space

$$G(\Omega) := \{f \in L^2(\Omega)^n; \exists p \in L^2_{\mathrm{loc}}(\Omega),\ f = \nabla p\}. \tag{2.1}$$

We also define the L^2-Sobolev space $W^{k,2}(\Omega) = \{u \in L^2(\Omega); D^\alpha u \in L^2(\Omega), |\alpha| \le k\}$, $W_0^{k,2}(\Omega) = \overline{C_0^\infty(\Omega)}^{||\cdot||_{W^{k,2}(\Omega)}} \subseteq W^{k,2}(\Omega)$ and $W^{k,2}_{0,\sigma}(\Omega) := \overline{C^\infty_{0,\sigma}(\Omega)}^{||u||_{W^{k,2}(\Omega)^n}}$. And define the following norm

$$||f||_{q,s;I} := (\int_I ||f||^s_{L^q(\Omega)} dt)^{\frac{1}{s}}, \tag{2.2}$$

in particular, for simplicity and convenience, we denote $||f||_{q,s;T} := ||f||_{q,s;(0,T)}$.

The following theorem collects some properties of the Stokes operator A.

Theorem 2.1. *([14],P128-P129,P133) Let $\Omega(\subseteq \mathbb{R}^3)$ be any domain,and let $A : D(A) \to L^2_\sigma(\Omega)$ be the Stokes operator for Ω. Then we have:*

1).A is positive self adjoint with dense domain $D(A) \subseteq L^2_\sigma(\Omega)$, $C^\infty_{0,\sigma}(\Omega) \subseteq D(A) \subseteq W^{1,2}_{0,\sigma}(\Omega)$. It holds $N(A) = \{u \in D(A); Au = 0\} = \{0\}$.

2).If Ω is bounded, then $D(A^{-1}) = R(A) = L^2_\sigma(\Omega)$, and A^{-1} is a bounded operator.

3).If Ω is a uniform C^2-domain or if $\Omega = \mathbb{R}^3$, then

$$D(A) = W^{1,2}_{0,\sigma}(\Omega) \cap W^{2,2}(\Omega)^3, Au = -\nu P\Delta u, \tag{2.3}$$

and

$$||\nabla^2 u||_2 + \nu^{-1}||\nabla p||_2 \le C(\nu^{-1}||Au||_2 + ||\nabla u||_2 + ||u||_2) \tag{2.4}$$

for all $u \in D(A)$. Here $\nabla^2 u := (D_j D_k)^n_{j,k=1} u = (D_j D_k u_l)^n_{j,k,l=1}$, and $C = C(\Omega) > 0$ is a constant, and $p \in L^2_{loc}(\Omega)$ is defined up to a constant by the equation: $-\nu\Delta u + \nabla p = f$.

4).If Ω is a bounded C^2-domain, then

$$D(A) = L^2_\sigma(\Omega) \cap W^{1,2}_0(\Omega)^3 \cap W^{2,2}(\Omega)^3, \tag{2.5}$$

and

$$||u||_{W^{2,2}(\Omega)} + \nu^{-1}||\nabla p||_2 \le C\nu^{-1}||Au||_2 \tag{2.6}$$

for all $u \in D(A)$. Here $C = C(\Omega) > 0$ is a constant, and $p \in L^2(\Omega)$ is defined up to a constant by the equation: $-\nu\Delta u + \nabla p = f$.

5). $R(A^{-\frac{1}{2}}) = D(A^{\frac{1}{2}}) = W^{1,2}_{0,\sigma}(\Omega)$ and $A^{\frac{1}{2}}$ be a positive self-adjoin operator, satisfying

$$\left\langle A^{\frac{1}{2}}u, A^{\frac{1}{2}}v \right\rangle = \nu \langle \nabla u, \nabla v \rangle, \quad ||A^{\frac{1}{2}}u||_2 = \nu^{\frac{1}{2}}||\nabla u||_2, \tag{2.7}$$

for all $u, v \in W^{1,2}_{0,\sigma}(\Omega)$.

6). If Ω is a bounded domain, then $D(A^{-\frac{1}{2}}) = R(A^{\frac{1}{2}}) = L^2_\sigma(\Omega)$, and $A^{-\frac{1}{2}}$ be a bounded positive self-adjoin operator with operator norm

$$||A^{-\frac{1}{2}}|| \le C\nu^{-\frac{1}{2}}. \tag{2.8}$$

Now let's denote $\langle u, v\rangle_{\Omega,T} := \langle u, v\rangle_{\Omega,(0,T)} := \int_0^T \langle u(t,\cdot), v(t,\cdot)\rangle\, dt := \int_0^T \mathrm{d}t \int_\Omega u(t,x)\cdot v(t,x)\mathrm{d}x$, define weak solution and strong solution of the Navier–Stokes Equations (1.1) as follow:

Definition 2.2. *Let $\Omega \subseteq \mathbb{R}^n, n = 3$, be any domain,and let $0 < T \le \infty, u^0 \in L^2_\sigma(\Omega)$ and let $f = f_0 + \mathrm{div}F$ with*

$$f_0 \in L^1_{loc}([0,T); L^2(\Omega)^n), F \in L^1_{loc}([0,T); L^2(\Omega)^{n\times n}). \tag{2.9}$$

Then

$$u \in L^\infty_{loc}([0,T); L^2_\sigma(\Omega)) \cap L^2_{loc}([0,T); W^{1,2}_{0,\sigma}(\Omega)), \tag{2.10}$$

is called a weak solution of the Navier–Stokes system (1.1) with data f, u^0, iff

$$-\langle u, v_t\rangle_{\Omega,T} + \nu\langle\nabla u, \nabla v\rangle_{\Omega,T} - \langle u\otimes u, \nabla v\rangle_{\Omega,T} = \left\langle u^0, v(0)\right\rangle + [f, v]_{\Omega,T} \tag{2.11}$$

holds for all $v\in C_0^\infty([0,T); C_{0,\sigma}^\infty(\Omega))$, where $[f,v]_{\Omega,T} := \langle f_0, v\rangle_{\Omega,T} - \langle F, \nabla v\rangle_{\Omega,T}$.

If u is such a weak solution, and p a distribution such that

$$u_t + u\cdot\nabla u = \nu\Delta u - \nabla p + f, \tag{2.12}$$

holds in the sense of distributions in $(0,T)\times\Omega$, then p is called an associated pressure of u.

A weak solution u of the Navier–Stokes system (1.1) is called a strong solution on $[0,T)$, if u satisfy the following Serrin's condition:

$$u\in L^s_{loc}([0,T); L^q(\Omega)^n), \qquad 0<T\le\infty, \tag{2.13}$$

with $n<q<\infty, 2<s<\infty$ such that

$$\frac{n}{q}+\frac{2}{s}\le 1. \tag{2.14}$$

The following is the expression formula for the weak solution of the Navier–Stokes system (1.1) with data f, u^0:

Theorem 2.3. *([14],P270) Let $\Omega\subseteq\mathbb{R}^n, n=3$, be any domain, let $0<T\le\infty, 1<s\le\frac{4}{n}$, let $u^0\in L^2_\sigma(\Omega), f=f_0+\mathrm{div}F$ with*

$$f_0\in L^1_{loc}([0,T); L^2(\Omega)^n), F\in L^s_{loc}([0,T); L^2(\Omega)^{n\times n}), \tag{2.15}$$

and let

$$u\in L^\infty_{loc}([0,T); L^2_\sigma(\Omega))\cap L^2_{loc}([0,T); W^{1,2}_{0,\sigma}(\Omega)). \tag{2.16}$$

Suppose u is a weak solution of the Navier–Stokes system (1.1) with data f, u^0. Then, after redefinition on a null set of $[0,T)$, $u:[0,T)\to L^2_\sigma(\Omega)$ is weakly continuous with $u(0)=u^0$,

$$\int_0^t S(t-\tau)A^{-\frac{1}{2}}P\mathrm{div}(F(\tau)-u(\tau)\otimes u(\tau))d\tau \in D(A^{\frac{1}{2}}), \tag{2.17}$$

and

$$\begin{aligned} u(t) = S(t)u^0 &+ \int_0^t S(t-\tau)Pf_0(\tau)d\tau \\ &+ A^{\frac{1}{2}}\int_0^t S(t-\tau)A^{-\frac{1}{2}}P\mathrm{div}(F(\tau)-u(\tau)\otimes u(\tau))d\tau, \end{aligned} \tag{2.18}$$

for all $t\in[0,T)$, where $S(t):=e^{-tA}, \quad t\ge 0$.

Conversely, let u satisfy the conditions(2.17) and (2.18) at least for almost all $t\in[0,T)$, then u is a weak solution of the Navier–Stokes system (1.1) with data f, u^0.

We know that, from the following theorem, there exists at least one weak solution u of (1.1) satisfying the energy inequality (2.21). Such weak solutions are called *Leray-Hopf solutions*:

Theorem 2.4. *([14],P320) Let $\Omega \subseteq \mathbb{R}^n, n = 2, 3,$ be any domain, let $0 < T \leq \infty, u^0 \in L^2_\sigma(\Omega)$, and $f = f_0 + \mathrm{div} F$ with*

$$f_0 \in L^1_{loc}([0,T); L^2(\Omega)^n), F \in L^2_{loc}([0,T); L^2(\Omega)^{n\times n}). \tag{2.19}$$

Then there exists a weak solution

$$u \in L^\infty_{loc}([0,T); L^2_\sigma(\Omega)) \cap L^2_{loc}([0,T); W^{1,2}_{0,\sigma}(\Omega)), \tag{2.20}$$

of the Navier–Stokes systems (1.1), satisfying the following properties:

a). $u : [0,T) \to L^2_\sigma(\Omega)$ is weakly continuous, after a redefinition on a null set of $[0,T]$.

b).

$$\frac{1}{2}||u(t)||_2^2 + \nu \int_0^t ||\nabla u||_2^2 d\tau \leq \frac{1}{2}||u^0||_2^2 + \int_0^t \langle f_0, u\rangle \, \mathrm{d}\tau - \int_0^t \langle F, \nabla u\rangle \, \mathrm{d}\tau \tag{2.21}$$

for all t with $0 \leq t < T$.

c).

$$\frac{1}{2}||u(t)||^2_{2,\infty;T'} + \nu||\nabla u||^2_{2,2;T'} \leq 2||u^0||_2^2 + 4\nu^{-1}||F||^2_{2,2;T'} + 8||f_0||^2_{2,1;T'} \tag{2.22}$$

for all T' with $0 \leq T' < T$.

Here is the existence theorem of associated pressure p of weak solutions u:

Theorem 2.5. *([14],P295) Let $\Omega \subseteq \mathbb{R}^n, n = 2, 3,$ be any domain, let $0 < T \leq \infty, u^0 \in L^2_\sigma(\Omega)$, $s = \frac{4}{3}$, and $f = f_0 + \mathrm{div} F$ with*

$$f_0 \in L^1_{loc}([0,T); L^2(\Omega)^n), F \in L^s_{loc}([0,T); L^2(\Omega)^{n\times n}), \tag{2.23}$$

and let

$$u \in L^\infty_{loc}([0,T); L^2_\sigma(\Omega)) \cap L^2_{loc}([0,T); W^{1,2}_{0,\sigma}(\Omega)), \tag{2.24}$$

be a weak solution of the Navier–Stokes systems (1.1) with data f, u^0.

Then there exists a function $\hat{p} \in L^s([0,T); L^2_{loc}(\Omega))$ such that the time derivative

$$p = \frac{\partial \hat{p}}{\partial t} = \hat{p}_t$$

is an associated pressure of $\widetilde{u}$.

If $n = 3$, we do not know whether the energy equality is satisfied for each weak solution u of (1.1). This is an open problem up to now. In order to get the energy equality we need addition condition, for example, (2.27) below:

Theorem 2.6. *([14],P272) Let* $\Omega \subseteq \mathbb{R}^n, n = 2, 3,$ *be any domain, let* $0 < T \leq \infty, u^0 \in L^2_\sigma(\Omega)$, *and* $f = f_0 + \mathrm{div} F$ *with*

$$f_0 \in L^1_{loc}([0,T); L^2(\Omega)^n), F \in L^2_{loc}([0,T); L^2(\Omega)^{n\times n}), \tag{2.25}$$

and let

$$u \in L^\infty_{loc}([0,T); L^2_\sigma(\Omega)) \cap L^2_{loc}([0,T); W^{1,2}_{0,\sigma}(\Omega)), \tag{2.26}$$

be a weak solution of the Navier–Stokes systems (1.1) with data f, u^0. *Suppose additionally that*

$$u \otimes u \in L^2(0,T; L^2(\Omega)^{n\times n}). \tag{2.27}$$

Then, after a redefinition on a null set, $u : [0,T) \to L^2_\sigma(\Omega)$ *is strongly continuous, we obtain the energy equality*

$$\frac{1}{2}||u(t)||_2^2 + \nu \int_0^t ||\nabla u||_2^2 d\tau = \frac{1}{2}||u^0||_2^2 + \int_0^t \langle f_0, u\rangle \, \mathrm{d}\tau - \int_0^t \langle F, \nabla u\rangle \, \mathrm{d}\tau, \tag{2.28}$$

for all $t \in [0,T)$, *and the inequality*

$$\frac{1}{2}||u(t)||^2_{2,\infty;T'} + \nu||\nabla u||^2_{2,2;T'} \leq 2||u^0||_2^2 + 4\nu^{-1}||F||^2_{2,2;T'} + 8||f_0||^2_{2,1;T'}, \tag{2.29}$$

for all T' *with* $0 \leq T' < T$.

In the following we will introduce the local existence of strong solution u of the Navier–Stokes systems (1.1):

Theorem 2.7. *([14],P345) Let* $\Omega \subseteq \mathbb{R}^3$, *be any three-dimensional domain, let* $0 < T \leq \infty$, $u^0 \in D(A^{\frac{1}{4}})$, *and let* $f = f_0 + \mathrm{div} F$ *with*

$$f_0 \in L^{\frac{4}{3}}(0,T; L^2(\Omega)^3), \quad F \in L^4(0,T; L^2(\Omega)^{3\times 3})$$

Then there exists a constant $K > 0$, *not depending on* Ω, u^0, f, T *and* ν, *with the following property:*

Choose any $T', 0 < T' \leq T$, *such that*

$$||f_0||_{2,\frac{4}{3};T'} + \nu^{-\frac{1}{2}}||F||_{2,4;T'} + ||(I - e^{-2T'A})A^{\frac{1}{4}}u^0||_2^{\frac{1}{8}}||A^{\frac{1}{4}}u^0||_2^{\frac{7}{8}} \leq K\nu^{\frac{5}{4}}. \tag{2.30}$$

Then in the interval $[0,T')$, *there exists a uniquely determined strong solution*

$$u \in L^\infty_{\mathrm{loc}}([0,T'); L^2_\sigma(\Omega)) \cap L^2_{\mathrm{loc}}([0,T'); W^{1,2}_{0,\sigma}(\Omega)), \tag{2.31}$$

of the Navier-Stokes system (1.1) with data u^0. *And the solution* u *satisfies Serrin's condition*

$$u \in L^8(0,T'; L^4(\Omega)^3) \tag{2.32}$$

with $\frac{3}{4} + \frac{2}{8} = 1$.

The following is a lemma for the local existence of strong solutions that is more convenient to apply than the previous one.

Lemma 2.8. *([14],P352) Let $K > 0$ be the constant as defined in the Theorem 2.7, and let $f = f_0 + \mathrm{div} F$ with*

$$f_0 \in L^{\frac{4}{3}}(0,T;L^2(\Omega)^3), \quad F \in L^4(0,T;L^2(\Omega)^{3\times 3}),$$

select one sufficiently small positive number $T_1 < T$, satisfy

$$||f_0||_{2,\frac{4}{3};T_1} + \nu^{-\frac{1}{2}}||F||_{2,4;T_1} + 4T_1^{\frac{1}{32}}(\sqrt{\nu}||\nabla u^0||_2 + ||u^0||_2) \leq K\nu^{\frac{5}{4}}. \tag{2.33}$$

Then on the interval $[0,T_1)$, the equation (1.1) exists a uniquely strong solution $u(t,x)$ satisfies Serrin's condition: $u(t,x) \in L^8(0,T_1;(L^4(\Omega))^3)$.

In the next step we will introduce the further regularity results of weak solution u of the Navier–Stokes systems (1.1), under the case of smooth exterior forces f:

Theorem 2.9. *([14],P300) Let $\Omega = \mathbb{R}^n$ or $\Omega \subseteq \mathbb{R}^n$, $n = 2,3$, be a uniform C^2-domain. Suppose Ω is also a C^∞-domain if $\Omega \neq \mathbb{R}^n$. Let $0 < T \leq \infty, u^0 \in W^{1,2}_{0,\sigma}(\Omega), f \in C_0^\infty(\overline{(0,T)\times\Omega})^n$, and let*

$$u \in L^\infty_{loc}([0,T);L^2_\sigma(\Omega)) \cap L^2_{loc}([0,T);W^{1,2}_{0,\sigma}(\Omega)), \tag{2.34}$$

be a weak solution of the Navier–Stokes systems (1.1) with data f, u^0. Assume additionally that

$$u \in L^s_{loc}([0,T);L^q(\Omega)^n), \tag{2.35}$$

with $n < q < \infty, 2 < s < \infty$ such that $\frac{n}{q} + \frac{2}{s} \leq 1$. Then, after a redefinition on a null set of in $[0,T)\times\Omega$, we obtain the weak solution u and associated pressure p satisfy

$$u \in C^\infty((0,T)\times\Omega)^n, \qquad p \in C^\infty((0,T)\times\Omega). \tag{2.36}$$

The following, let's introduce continuous embeddings of the domain $D(A^\alpha)$ into certain L^q-spaces.

Lemma 2.10. *([14],P142) Let $\Omega \subseteq \mathbb{R}^n$,$n \geq 2$, be any domain, let $0 \leq \alpha \leq \frac{1}{2}, 2 \leq q < \infty$ with*

$$2\alpha + \frac{n}{q} = \frac{n}{2}, \tag{2.37}$$

and let A be the Stokes operator for Ω.

Then $u \in D(A^\alpha)$ implies $u \in L^q(\Omega)^n$ and

$$||u||_{L^q(\Omega)^n} \leq C\nu^{-\alpha}||A^\alpha u||_{L^2(\Omega)^n}, \tag{2.38}$$

where $C = C(\alpha,q) > 0$ is a constant.

Lemma 2.11. *([14],P265-P266) Let $\Omega \subseteq \mathbb{R}^n$,$n=3$, be any domain, let $0<T\leq\infty$, and let*

$$u, v \in L^\infty_{\text{loc}}([0,T);L^2_\sigma(\Omega)) \cap L^2_{\text{loc}}([0,T);W^{1,2}_{0,\sigma}(\Omega)) \tag{2.39}$$

then we have

a) $|v(t)||\nabla u(t)||u(t)|,\ |v(t)||\nabla|u(t)|^2|,\ |\nabla v(t)||u(t)|^2\ \in L^1(\Omega)$ *and*

$$\begin{aligned}\langle v(t)\cdot\nabla u(t), u(t)\rangle &= \langle \operatorname{div}(v(t)\otimes u(t)), u(t)\rangle \\ &= -\langle v(t)\otimes u(t), \nabla u(t)\rangle = -\frac{1}{2}\left\langle v(t), \nabla|u(t)|^2\right\rangle \\ &= \frac{1}{2}\left\langle \text{div } v(t), |u(t)|^2\right\rangle = 0,\end{aligned} \tag{2.40}$$

for almost all $t\in[0,T)$.

b)

$$\begin{aligned}||u||_{q',s';T'} &\leq C\nu^{-\frac{1}{s'}}||A^{\frac{1}{s'}}u||_{2,s';T'} \\ &\leq C\nu^{-\frac{1}{s'}}||A^{\frac{1}{2}}u||^{\frac{2}{s'}}_{2,2;T'}||u||^{1-\frac{2}{s'}}_{2,\infty;T'} \\ &\leq C\nu^{-\frac{1}{s'}}(||A^{\frac{1}{2}}u||_{2,2;T'}+||u||_{2,\infty;T'}) \\ &\leq C'\nu^{-\frac{1}{s'}}E_{T'}(u)^{\frac{1}{2}} < +\infty\end{aligned} \tag{2.41}$$

with $2\leq q'<+\infty,\ 2\leq s'\leq+\infty$ satisfying

$$\frac{n}{q'}+\frac{2}{s'}=\frac{n}{2}, \tag{2.42}$$

and with $0<T'<T; C=C(s',n)>0, C'=C'(s',n)>0$.

Lemma 2.12. *([14],P55) Let $\Omega\subseteq\mathbb{R}^n$ be an arbitrary domain with $n\geq 2$. Then there exists a sequence $(\Omega_j)_{j=1}^\infty$ of bounded Lipschitz subdomains of Ω and a sequence $(\varepsilon_j)_{j=1}^\infty$ of positive numbers with the following properties:*

a) $\overline{\Omega_j}\subseteq\Omega_{j+1}, j\in\mathbb{N}$,

b) *dist* $(\partial\Omega_{j+1},\Omega_j)\geq\varepsilon_{j+1},, j\in\mathbb{N}$,

c) $\lim\limits_{j\to+\infty}\varepsilon_j=0$,

d) $\Omega=\bigcup_{j=1}^\infty\Omega_j$.

Consider the following linear incompressible Stokes equations in three spatial dimensions with external force f:

$$\begin{cases} u_t=\nu\Delta u-\nabla p+f, & t>0, x\in\Omega \\ \text{div } u=0, & t\geq 0, x\in\Omega \\ u(0,x)=u^0(x), & x\in\Omega \\ u(t,x)=0, & t\geq 0, x\in\partial\Omega. \end{cases} \tag{2.43}$$

Where $\Omega\subseteq\mathbb{R}^n, (n=3)$ is an arbitrary domain. The boundary condition $u|_{\partial\Omega}=0$ is omitted if $\Omega=\mathbb{R}^3$, the initial solenoidal functions $u^0(x)$ satisfying compatibility conditions: $u^0|_{\partial\Omega}=0$ is given, and ν is a positive coefficient (the viscosity).

Definition 2.13. *([14],P220) Let* $\Omega \subseteq \mathbb{R}^n, n = 3$*, be any domain,and let* $0 < T \leq \infty, u^0 \in L^2_\sigma(\Omega)$ *and let* $f = f_0 + \mathrm{div} F$ *with*

$$f_0 \in L^1_{loc}([0,T); L^2(\Omega)^n), F \in L^1_{loc}([0,T); L^2(\Omega)^{n\times n}). \tag{2.44}$$

Then a function

$$u \in L^1_{loc}([0,T); W^{1,2}_{0,\sigma}(\Omega)), \tag{2.45}$$

is called a weak solution of the Stokes system (2.43) with data f, u^0, *iff*

$$-\langle u, v_t\rangle_{\Omega,T} + \nu \langle \nabla u, \nabla v\rangle_{\Omega,T} = \langle u^0, v(0)\rangle + [f, v]_{\Omega,T} \tag{2.46}$$

holds for all $v \in C^\infty_0([0,T); C^\infty_{0,\sigma}(\Omega))$*, where* $[f,v]_{\Omega,T} := \langle f_0, v\rangle_{\Omega,T} - \langle F, \nabla v\rangle_{\Omega,T}$.

A distribution p *in* $(0,T)\times\Omega$ *is called an associated pressure of a weak solution* u*, iff*

$$u_t = \nu\Delta u - \nabla p + f, \tag{2.47}$$

holds in the sense of distributions.

In the following we will introduce these equations the theory of properties of solutions. The linear theory considered in this chapter which will be used later.

Theorem 2.14. *([14],P226) Let* $\Omega \subseteq \mathbb{R}^n, n \geq 2$*, be any domain, let* $0 < T \leq \infty, u^0 \in L^2_\sigma(\Omega)$*, and let* $f = f_0 + \mathrm{div}\ F$ *with*

$$f_0 \in L^1(0,T; L^2(\Omega)^n), \quad F \in L^2(0,T; L^2(\Omega)^{n\times n})$$

and let

$$u \in L^1_{loc}([0,T); W^{1,2}_{0,\sigma}(\Omega))$$

is a weak solution of the Stokes system (2.43) with data f, u^0.

Then u *has the following properties:*

a)

$$u \in L^\infty(0,T; L^2_\sigma(\Omega)), \quad \nabla u \in L^2(0,T; L^2(\Omega)^{n\times n}). \tag{2.48}$$

b) $u : [0,T) \to L^2_\sigma(\Omega)$ *is strongly continuous, after redefining on a null set of* $[0,T)$*,* $u(0) = u^0$*, and the energy equality*

$$\frac{1}{2}||u(t)||_2^2 + \nu\int_0^t ||\nabla u||_2^2 \mathrm{d}\tau = \frac{1}{2}||u^0||_2^2 + \int_0^t < f_0, u > \mathrm{d}\tau - \int_0^t < F, \nabla u > \mathrm{d}\tau, \tag{2.49}$$

holds for all $t \in [0,T)$.

c)

$$\frac{1}{2}||u||^2_{2,\infty;T} + \nu||\nabla u||^2_{2,2;T} \leq 2||u^0||_2^2 + 8||f_0||^2_{2,1;T} + \frac{4}{\nu}||F||^2_{2,2;T}. \tag{2.50}$$

Theorem 2.15. *([14],P238-P239) Let* $\Omega \subseteq \mathbb{R}^n, n \geq 2$*, be any domain, let* $0 < T \leq \infty, 1 < s \leq \rho < \infty, \frac{1}{2} + \frac{1}{\rho} \geq \frac{1}{s}$ *and that*

$$u^0 \in D(A^{1-\frac{1}{s}}) \quad \text{if} \quad s \geq 2, \quad u^0 \in D(A^{\frac{1}{2}}) = W^{1,2}_{0,\sigma}(\Omega) \quad \text{if} \quad 1 < s < 2.$$

Let

$$u \in L^1_{loc}([0,T); W^{1,2}_{0,\sigma}(\Omega))$$

be a weak solution of the Stokes system (2.43) with data $f = 0$ *and* u^0*.*

Then, after redefining u *on a null set of* $[0,T), u : [0,T) \to L^2_\sigma(\Omega)$ *is strongly continuous,* $u(0) = u^0$*, and*

$$u(t) = S(t)u^0,$$

for all $t \in [0,T)$*.*

Moreover, u *has the following properties:*

a)

$$||u'||_{2,s;T} + ||Au||_{2,s;T} \leq \begin{cases} 2||A^{1-\frac{1}{s}}u^0||_2, & \text{if} \quad s \geq 2, \\ C(||u^0||_2 + ||A^{\frac{1}{2}}u^0||_2), & \text{if} \quad 1 < s < 2 \end{cases} \tag{2.51}$$

with $C = C(s) > 0,\ u' + Au = 0$ *in* $L^s(0,T; L^2_\sigma(\Omega))$ *and therefore*

$$u'(t) + Au(t) = 0 \tag{2.52}$$

for almost all $t \in [0,T)$*.*

b)

$$\frac{1}{2}||u||^2_{2,\infty;T} + \nu||\nabla u||^2_{2,2;T} \leq 2||u^0||^2_2. \tag{2.53}$$

Theorem 2.16. *([14],P240-P241) Let* $\Omega \subseteq \mathbb{R}^n, n \geq 2$*, be any domain, let* $0 < T \leq \infty, 1 \leq s < \infty$*, and let* $f \in L^s(0,T; L^2(\Omega)^n)$*.*

Suppose

$$u \in L^1_{loc}([0,T); W^{1,2}_{0,\sigma}(\Omega)).$$

is a weak solution of the Stokes system (2.43) with data f *and* $u^0 = 0$ *.*

Then, after redefining u *on a null set of* $[0,T)$*, we get*

$$u(t) = (\mathcal{J}Pf)(t) = \int_0^t S(t-\tau)Pf(\tau)\mathrm{d}\tau \tag{2.54}$$

for all $t \in [0,T), u : [0,T) \to L^2_\sigma(\Omega)$ *is strongly continuous and* $u(0) = 0$*.*

Moreover, u *has the following properties:*

a)

$$||u'||_{2,s;T} + ||Au||_{2,s;T} \leq C||f||_{2,s;T} \tag{2.55}$$

with $1 < s < \infty, C = C(s) > 0$ *and*

$$u'(t) + Au(t) = Pf(t) \tag{2.56}$$

holds for almost all $t \in [0, T)$.

b)

$$\frac{1}{2}||u||^2_{2,\infty;T} + \nu||\nabla u||^2_{2,2;T} \leq 8||f||^2_{2,1;T}, \quad \textit{if} \quad s = 1. \tag{2.57}$$

c)

$$||u'||^2_{2,2;T} + \frac{1}{2}||A^{\frac{1}{2}}u||^2_{2,\infty;T} + ||Au||^2_{2,2;T} \leq 14||f||^2_{2,2;T}, \quad \textit{if} \quad s = 2. \tag{2.58}$$

Theorem 2.17. *([14],P243) Let* $\Omega \subseteq \mathbb{R}^n, n \geq 2$, *be any domain, let* $0 < T \leq \infty, 1 < s \leq \rho < \infty$, *and let* $F \in L^s(0, T; L^2(\Omega)^{n\times n})$.

Suppose

$$u \in L^1_{loc}([0, T); W^{1,2}_{0,\sigma}(\Omega))$$

is a weak solution of the Stokes system (2.43) with data $f = \mathrm{div}F$ *and* $u^0 = 0$.

Then we get

$$u(t) = A^{\frac{1}{2}} \int_0^t S(t-\tau)A^{-\frac{1}{2}}P\mathrm{div}F(\tau)\mathrm{d}\tau, \quad u(t) \in D(A^{-\frac{1}{2}}) \tag{2.59}$$

for almost all $t \in [0, T)$. *After redefining* u *on a null set of* $[0, T)$, $A^{-\frac{1}{2}}u : [0, T) \to L^2_\sigma(\Omega)$ *is strongly continuous and* $(A^{-\frac{1}{2}}u)(0) = 0$.

Moreover, u *has the following properties:*

a)

$$||(A^{-\frac{1}{2}}u)'||_{2,s;T} + ||A^{\frac{1}{2}}u||_{2,s;T} \leq C\nu^{-\frac{1}{2}}||F||_{2,s;T} \tag{2.60}$$

with $C = C(s) > 0$,

$$(A^{-\frac{1}{2}}u)'(t) + A^{\frac{1}{2}}u(t) = A^{-\frac{1}{2}}P\mathrm{div}F(t) \tag{2.61}$$

for almost all $t \in [0, T)$,*and* $A^{-\frac{1}{2}}u$ *is a weak solution of the Stokes system (2.43) with data* $A^{-\frac{1}{2}}P\mathrm{div}F$ *and* $u^0 = 0$.

b)

$$\frac{1}{2}||u||^2_{2,\infty;T} + ||A^{\frac{1}{2}}u||^2_{2,2;T} \leq 4\nu^{-1}||F||^2_{2,2;T}, \quad \textit{if} \quad s = 2. \tag{2.62}$$

3 The uniqueness of Leray-Hopf solution for a Class of incompressible Navier-Stokes Equations in three spatial dimensions

In this section, let $\Omega = \mathbb{R}^3$ or $\Omega \subset \mathbb{R}^3$ be an open nonempty connected bounded uniform C^2-domain with boundary $\partial\Omega$, and Ω is also a C^∞-domain. The main result of this chapter is the following.

Theorem 3.1. *Let the external force $f \in C_0^\infty([0,\infty); C_0^\infty(\Omega)^3)$ and the initial data $u^0 \in W_{0,\sigma}^{1,2}(\Omega)$. Then, for any $0 < T < \infty$, there exists a uniquely determined Leray-Hopf solutions*

$$\widetilde{u} \in L^\infty(0,T; L_\sigma^2(\Omega)) \cap L^2(0,T; W_{0,\sigma}^{1,2}(\Omega))$$

of the Navier-Stokes system (1.1) which satisfies the following energy inequality:

a)

$$\frac{1}{2}\|\widetilde{u}(t)\|_2^2 + \nu \int_0^t \|\nabla\widetilde{u}\|_2^2 \mathrm{d}\tau \le \frac{1}{2}\|u^0\|_2^2 + \int_0^t < f, \widetilde{u} > \mathrm{d}\tau, \qquad 0 \le t \le T. \tag{3.1}$$

and inequality:

b)

$$\frac{1}{2}||\widetilde{u}||_{2,\infty;T}^2 + \nu||\nabla\widetilde{u}||_{2,2;T}^2 \le 2\|u^0\|_2^2 + 8||f||_{2,1;T}^2. \tag{3.2}$$

Further, there exists a function $\hat{p} \in L^{\frac{4}{3}}([0,T); L_{loc}^2(\Omega))$ such that the time derivative

$$p = \frac{\partial\hat{p}}{\partial t} = \hat{p}_t$$

is an associated pressure of $\widetilde{u}$.

Before proving Theorem 3.1, let's introduce the following concepts and lemmas:
Let's define the (energy) equation:

$$E_T(u) := \frac{1}{2}(\operatorname*{esssup}_{t\in[0,T)} ||u(t)||_2)^2 + \nu \int_0^T ||\nabla u||_2^2 d\tau, \tag{3.3}$$

for $0 < T < +\infty$.

By the Theorem 2.4, we know that there exists at least one *Leray-Hopf solution* $\widetilde{u}(t,x;f,u^0)$ of (1.1) (For brevity, let's denote $\widetilde{u}(t,x;f,u^0)$ as $\widetilde{u}(t,x)$ or even $\widetilde{u}$). Recall, this means that $\widetilde{u} \in L^\infty(0,T; L_\sigma^2(\Omega)) \cap L^2(0,T; W_{0,\sigma}^{1,2}(\Omega))$, and the following (energy) quantity hold

$$E_T(\widetilde{u}) \le 2\|u^0\|_2^2 + 8||f||_{2,1;T}^2, \qquad ||\widetilde{u}(t)||_2 \le 2\|u^0\|_2 + 4||f||_{2,1;T}, \tag{3.4}$$

for $0 < t \le T < +\infty$.

Lemma 3.2. *Let $K > 0$ be the constant as defined in the Theorem 2.7, let the external force $f \in C_0^\infty([0,\infty); C_0^\infty(\Omega)^3)$, the initial data $u^0 \in W_{0,\sigma}^{1,2}(\Omega)$, and let function $\widetilde{u}(t,x;f,u^0)$ be one Leray-Hopf solution of (1.1), assume $||\nabla\widetilde{u}(a)||_2 < +\infty$ with some $a \in [0,\infty)$, select one sufficiently small positive $T_1 = T_1(a) < 1$, satisfy*

$$T_1^{\frac{1}{32}}(||f||_{2,2;\infty} + 4(\sqrt{\nu}||\nabla\widetilde{u}(a)||_2 + ||\widetilde{u}(a)||_2)) \le K\nu^{\frac{5}{4}}/2. \tag{3.5}$$

Then on the interval $[a, a+T_1]$, the equation (1.1) exists a uniquely strong solution $\widetilde{u}(t,x;f,\widetilde{u}(a))$ satisfies Serrin's condition: $\widetilde{u} \in L^8(a, a+T_1; (L^4(\Omega))^3)$.

Proof. Noting $T_1^{\frac{1}{4}} < T_1^{\frac{1}{32}}$ since $0 < T_1 < 1$, by the notational convention (2.2), it follows that

$$||f||_{2,\frac{4}{3};(a,a+T_1)} \leq ||f||_{2,2;(a,a+T_1)} T_1^{\frac{1}{4}} \leq ||f||_{2,2;\infty} T_1^{\frac{1}{32}}.$$

Applying the formula (2.33) of Lemma 2.8, we can easily prove the conclusion. □

We do not know whether *Leray-Hopf solutions* are regular for $n \geq 3$. If *Leray-Hopf solutions* develop singularity, it is natural to estimate the size of the singularities. Many mathematicians have done work in this area, for example: J.Leray([9]), Scheffer([15, 16, 17]), Caffarelli, L., Kohn, R., and Nirenberg, L.([3, 10]. Here, we will introduce time singularity only in formulation of [6].

Lemma 3.3. *Let $0 < T < \infty$, then there is a closed set $E_{\widetilde{u}}$ of Lebesgue measure zero in $(0, T)$ such that $\widetilde{u}$ is smooth in $((0, T) - E_{\widetilde{u}}) \times \Omega$.*

Proof. Let's define the following set

$$E_{\widetilde{u}} := \{t \in (0, T); ||\nabla \widetilde{u}(t)||_2 = +\infty\}. \tag{3.6}$$

Then, by the (energy) quantity (3.4), $E_{\widetilde{u}}$ is a set of Lebesgue measure zero. Using Lemma 3.2 and Theorem 2.9, applying the same method of ([6], Lemma 7.4), we can easily prove $\widetilde{u}$ is smooth in $((0, T) - E_{\widetilde{u}}) \times \Omega$. The proof is completed. □

Lemma 3.4. *Let $0 < T < \infty$, and let $E_{\widetilde{u}}$ be the set as defined in (3.6), then the function $\widetilde{u}$: $(0, T) - E_{\widetilde{u}} \to L^2_\sigma(\Omega)$ is strongly continuous.*

Proof. Let $(c_i, d_i), i \in I \subseteq \mathbb{N}$, be the connected components of $(0, T) - E_{\widetilde{u}}$. Noting that $E_{\widetilde{u}}$ be a closed set of Lebesgue measure zero in $(0, T)$ and $(0, T) - E_{\widetilde{u}} = \bigcup_{i \in I} \{(c_i, d_i)\}$ be an open set, for any $t_c \in (0, T) - E_{\widetilde{u}}$, it follows that there exists $i_0 \in I$, such that $t_c \in (c_{i_0}, d_{i_0})$. We just need to prove that $\widetilde{u} : (0, T) - E_{\widetilde{u}} \to L^2_\sigma(\Omega)$ is strongly continuous at $t = t_c$.

Select one sufficient small positive τ to make $[t_c - \tau, t_c + \tau] \subset (c_{i_0}, d_{i_0})$ hold, then by Lemma 3.3, it follows that $\widetilde{u} \in C^1([t_c - \tau, t_c + \tau] \times \Omega)^3$. Noting $\widetilde{u}|_{\partial\Omega} = 0$, it is clear that

$$C(\widetilde{u}) := \sup_{(t,x) \in [t_c - \tau, t_c + \tau] \times \Omega} |\widetilde{u}(t, x)| < +\infty. \tag{3.7}$$

From the above formula (3.7) and (3.4), we can get $\widetilde{u}$ satisfies the following Serrin's condition:

$$\begin{aligned} ||\widetilde{u}||_{q,s;[t_c-\tau,t_c+\tau]} &= (\int_{t_c-\tau}^{t_c+\tau} (\int_\Omega |\widetilde{u}(t, x)|^4 dx)^2 dt)^{\frac{1}{8}} \\ &\leq C^{\frac{1}{2}}(\widetilde{u})(\int_{t_c-\tau}^{t_c+\tau} (\int_\Omega |\widetilde{u}(t, x)|^2 dx)^2 dt)^{\frac{1}{8}} \\ &\leq C^{\frac{1}{2}}(\widetilde{u})(2||u^0||_2 + 4||f||_{2,1;T})^{\frac{1}{2}} (2\tau)^{\frac{1}{8}} < +\infty, \end{aligned} \tag{3.8}$$

with $\frac{3}{q} + \frac{2}{s} = 1,\ s = 8,\ q = 4$. Using Hölder inequality and b) of Lemma 2.11 with $s' = \frac{8}{3}, q' = 4, \frac{n}{q'} + \frac{2}{s'} = \frac{n}{2}, n = 3$. The above formula leads to

$$\begin{aligned} ||\widetilde{u} \otimes \widetilde{u}||_{2,2;[t_c-\tau,t_c+\tau]} &\leq C||\widetilde{u}||_{q,s;[t_c-\tau,t_c+\tau]} ||\widetilde{u}||_{q',s';[0,T)} \\ &\leq C' \nu^{-\frac{1}{s'}} ||\widetilde{u}||_{q,s;[t_c-\tau,t_c+\tau]} E_T^{\frac{1}{2}}(\widetilde{u}) < \infty, \end{aligned} \tag{3.9}$$

with absolute constant $C>0, C'>0$.

Consider the following incompressible Navier-Stokes equations on $[t_c-\tau, t_c+\tau]$:

$$
\begin{cases}
u_t + Au + Pu\cdot\nabla u = Pf, & t_c-\tau\le t\le t_c+\tau, x\in\Omega,\\
u(t_c-\tau, x) = \widetilde{u}(t_c-\tau, x), & x\in\Omega,\\
u(t,x)=0, & t_c-\tau\le t\le t_c+\tau, x\in\partial\Omega,
\end{cases}
\tag{3.10}
$$

where P be the Helmholtz projection operator from $L^2(\Omega)^3$ onto $L^2_\sigma(\Omega)$, $A=-\nu P\Delta$ be the Stokes operator and the symbol $Pu\cdot\nabla u := P(u\cdot\nabla u) := P((u\cdot\nabla)u)$.

Then, there exists a determined weak solution

$$\widetilde{u}\in L^\infty(t_c-\tau, t_c+\tau; L^2_\sigma(\Omega))\cap L^2(t_c-\tau, t_c+\tau; W^{1,2}_{0,\sigma}(\Omega))$$

of the Navier–Stokes systems (3.10) which satisfies the Serrin's condition (3.8). Thus, the weak solution $\widetilde{u}$ be uniquely determined strong solution of the Navier–Stokes systems (3.10) on the interval $[t_c-\tau, t_c+\tau]$, with data $f, \widetilde{u}(t_c-\tau, x)$. Noting the formula (3.9) hold, applying the Theorem 2.6, replacing $(0,T)$ with $(t_c-\tau, t_c+\tau)$, we can obtain: the weak solution $\widetilde{u}:[t_c-\tau, t_c+\tau]\to L^2_\sigma(\Omega)$ is strongly continuous. Thus, the weak solution $\widetilde{u}:[t_c-\tau, t_c+\tau]\to L^2_\sigma(\Omega)$ is strongly continuous at $t=t_c$. The proof is complete. □

The following, we prove some important properties of the following linear nonhomogeneous incompressible Navier-Stokes systems:

$$
\begin{cases}
w_t + Aw - P\overline{u}\cdot\nabla w - L(\overline{w})w = Pg, & 0\le t\le T<+\infty, x\in\Omega\\
w(0,x)=\psi^0(x), & x\in\Omega\\
w(t,x)=0, & t\ge 0, x\in\partial\Omega,
\end{cases}
\tag{3.11}
$$

where $A=-\nu P\Delta$ is the Stokes operator, P is a Helmholtz projection operator, $P\overline{u}\cdot\nabla w := P(\overline{u}\cdot\nabla w) := P((\overline{u}\cdot\nabla)w)$, and $L(\overline{w}): W^{1,2}_{0,\sigma}(\Omega)\to L^2_\sigma(\Omega)$ is a linear operator satisfying

$$\langle L(\overline{w})w, u\rangle := \langle\nabla w, u\otimes\overline{w}\rangle, \text{ for } u\in L^2_\sigma(\Omega),\ 0\le t\le T<+\infty, \tag{3.12}$$

the functions $\overline{u}\in L^\infty(0,T;L^2_\sigma(\Omega))\cap L^2(0,T;W^{1,2}_{0,\sigma}(\Omega))$, $\overline{w}\in L^\infty((0,T)\times\Omega)^3\cap L^\infty(0,T;L^2_\sigma(\Omega))$, the external force g and the initial solenoidal function $\psi^0(x)$ with $\psi^0|_{\partial\Omega}=0$ are given.

In order to express our ideas clearly and use the theory of operator semigroup, let's define the following convolution operator $\mathcal{J}$:

$$(\mathcal{J}g)(t) := \int_0^t S(t-\tau)g(\tau)d\tau,\quad S(t):=e^{-tA},\quad t\ge 0. \tag{3.13}$$

We have the following explicit representation formula:

Lemma 3.5. *Let $0<T<\infty, g\in L^1(0,T;L^2(\Omega)^3), \psi^0\in L^2_\sigma(\Omega)$ satisfy compatibility conditions: $\psi^0|_{\partial\Omega}=0$, and let*

$$w\in L^\infty(0,T;L^2_\sigma(\Omega))\cap L^2(0,T;W^{1,2}_{0,\sigma}(\Omega)). \tag{3.14}$$

Suppose w is a weak solution of the Navier–Stokes system (3.11) with data g, ψ^0. Then, after redefinition on a null set of $[0,T]$, $w : [0,T] \to L^2_\sigma(\Omega)$ is weakly continuous with $w(0) = \psi^0$,

$$(\mathcal{J}A^{-\frac{1}{2}}P\mathrm{div}(\overline{u}\otimes w))(t) \in D(A^{\frac{1}{2}}), \tag{3.15}$$

and

$$w(t) = S(t)\psi^0 + (\mathcal{J}Pg)(t) + (\mathcal{J}L(\overline{w})w)(t) + A^{\frac{1}{2}}(\mathcal{J}A^{-\frac{1}{2}}P\mathrm{div}(\overline{u}\otimes w))(t), \tag{3.16}$$

for all $t \in [0,T]$, where the matrix $\overline{u}\otimes w$ means the usual tensor product and $\mathrm{div}(\overline{u}\otimes w) = D_1(\overline{u}_1 w) + D_2(\overline{u}_2 w) + D_3(\overline{u}_3 w) = \overline{u}\cdot\nabla w$.

Conversely, let w satisfy the conditions (3.15) and (3.16) at least for almost all $t \in [0,T]$, then w is a weak solution of the Navier–Stokes system (3.11) with data g, ψ^0.

Proof. Applying the same method of ([14], Theorem 1.3.1 of P270-P271), replacing $u\cdot\nabla u,\ f_0$ by $\overline{u}\cdot\nabla w,\ g + L(\overline{w})w$, we can easily prove the conclusion. □

Here is the existence of (at least one) *Leray-Hopf solution* of the Navier-Stokes systems (3.11).

Lemma 3.6. *Let $0 < T < \infty, g \in L^1(0,T; L^2(\Omega)^3), \psi^0 \in L^2_\sigma(\Omega)$ satisfy compatibility conditions: $\psi^0|_{\partial\Omega} = 0$, Then, there exists a determined weak solution*

$$w \in L^\infty(0,T; L^2_\sigma(\Omega)) \cap L^2(0,T; W^{1,2}_{0,\sigma}(\Omega)), \tag{3.17}$$

of the Navier–Stokes systems (3.11) satisfying the following properties:

a) $w : [0,T] \to L^2_\sigma(\Omega)$ is weakly continuous, after a redefinition on a null set of in $[0,T]$ with $w(0) = \psi^0$.

b) $(\mathcal{J}A^{-\frac{1}{2}}P\mathrm{div}(\overline{u}\otimes w))(t) \in D(A^{\frac{1}{2}})$ and

$$w(t) = S(t)\psi^0 + (\mathcal{J}Pg)(t) + (\mathcal{J}L(\overline{w})w)(t) + A^{\frac{1}{2}}(\mathcal{J}A^{-\frac{1}{2}}P\mathrm{div}(\overline{u}\otimes w))(t) \tag{3.18}$$

for all $t \in [0,T]$.

c)

$$\frac{1}{2}||w(t)||_2^2 + \nu\int_0^t ||\nabla w||_2^2 \mathrm{d}\tau \le \frac{1}{2}||\psi^0||_2^2 + \int_0^t \langle g, w\rangle\, d\tau + \int_0^t \langle \nabla w, w\otimes\overline{w}\rangle\, \mathrm{d}\tau, \tag{3.19}$$

for all $t \in [0,T]$.

d)

$$\frac{1}{2}||w||^2_{2,\infty;T} + \nu||\nabla w||^2_{2,2;T} \le C_0(\overline{w},\nu,T)||\psi^0||_2^2 + C_0(\overline{w},\nu,T)||g||^2_{2,1;T}, \tag{3.20}$$

where $C_0(\overline{w},\nu,T)$ be a constant.

Proof. See Section 4. □

The following uniqueness result of the Navier-Stokes systems (3.11) is parallel to Serrin and Masuda's Theorem ([18] or [14], P276):

Lemma 3.7. *Let* $n = 3$, $0 < T < \infty, g \in L^1(0,T;L^2(\Omega)^3), \psi^0 \in L^2_\sigma(\Omega)$ *satisfy compatibility conditions:* $\psi^0|_{\partial\Omega} = 0$, *and let*

$$u, w \in L^\infty(0,T;L^2_\sigma(\Omega)) \cap L^2(0,T;W^{1,2}_{0,\sigma}(\Omega)), \tag{3.21}$$

be two Leray-Hopf solutions of the Navier–Stokes systems (3.11) with same data g, ψ^0. *Suppose additionally that the function* $\overline{u}$ *satisfies the following Serrin's condition:*

$$\overline{u} \in L^s(0,T;L^q(\Omega)^n) \tag{3.22}$$

with $n < q < \infty, 2 < s < \infty$, *such that*

$$\frac{n}{q} + \frac{2}{s} \leq 1, \tag{3.23}$$

Then $u = w$ *in* $[0,T]$.

Proof. Without loss of generality we may assume that (3.23) holds with $\frac{n}{q} + \frac{2}{s} = 1$. In fact, if $\frac{n}{q} + \frac{2}{s} < 1$, we can choose $2 < s_1 < s$, so that $\frac{n}{q} + \frac{2}{s_1} = 1$. Then, we choose $\gamma > 1$ with $\frac{1}{s_1} = \frac{1}{s} + \frac{1}{\gamma}$, and with Hölder's inequality we get

$$||\overline{u}||_{q,s_1;T} \leq T^{\frac{1}{\gamma}} ||\overline{u}||_{q,s;T} < \infty,$$

and formula (3.23) holds with s replaced by s_1. Thus we may assume that (3.23) holds with $\frac{n}{q} + \frac{2}{s} = 1$. The following, let's choose s', q', such that $\frac{1}{2} = \frac{1}{s} + \frac{1}{s'}, \frac{1}{2} = \frac{1}{q} + \frac{1}{q'}$. Then, it follows that $\frac{n}{q'} + \frac{2}{s'} = \frac{n}{2}$ with $n = 3$. Now, by Hölder inequality and d) of the Lemma 3.6, using Lemma 2.11 with $s' = \frac{8}{3}, q' = 4, \frac{n}{q'} + \frac{2}{s'} = \frac{n}{2}, n = 3$. We can get

$$\begin{aligned} ||\overline{u} \otimes w||_{2,2;T} &\leq C||\overline{u}||_{q,s;T}||w||_{q',s';T} \\ &\leq C'\nu^{-\frac{1}{s'}}||\overline{u}||_{q,s;T}E_T(w)^{\frac{1}{2}} < \infty, \end{aligned} \tag{3.24}$$

with absolute constant $C > 0, C' > 0$.

In addition, it is clear that function $\phi := u - w$ satisfy the following linear incompressible equation:

$$\begin{cases} \phi_t + A\phi - P\overline{u} \cdot \nabla\phi - L(\overline{w})\phi = 0, & 0 \leq t \leq T, x \in \Omega \\ \phi(0,x) = 0, & x \in \Omega \\ \phi(t,x) = 0, & t \geq 0, x \in \partial\Omega. \end{cases} \tag{3.25}$$

Applying the same method as above formula (3.24), using the above formula (3.22), Hölder inequality and Lemma 2.11, it is easy to prove that the function ϕ satisfy

$$L(\overline{w})\phi \in L^2(0,T;L^2(\Omega)^3), \quad \overline{u} \otimes \phi \in L^2(0,T;L^2(\Omega)^{3\times 3}). \tag{3.26}$$

Observe that ϕ is also a weak solution of the following linear system

$$\phi_t - \nu\Delta\phi + \nabla p = \tilde{f}, \quad \mathrm{div}\phi = 0, \quad \phi|_{\partial\Omega} = 0, \quad \phi(0) = 0$$

with $\tilde{f} = L(\overline{w})\phi + \mathrm{div}(\overline{u} \otimes \phi)$. Apply Theorem 2.14, this shows that ϕ is strongly continuous, after a corresponding redefinition, and that

$$\frac{1}{2}||\phi(t)||_2^2 + \nu \int_0^t ||\nabla\phi||_2^2 \mathrm{d}\tau = \int_0^t \langle \overline{u} \otimes \phi, \nabla\phi \rangle \,\mathrm{d}\tau + \int_0^t \langle \nabla\phi, \phi \otimes \overline{w} \rangle \,\mathrm{d}\tau, \tag{3.27}$$

for all $t \in [0, T]$.

According to the above formula (3.37), applying the Lemma 2.11, combining formula (3.27), we can obtain that

$$\begin{aligned} \sup_{0\le\tau\le t} (\frac{1}{2}||\phi(\tau)||_2^2 + \nu \int_0^\tau ||\nabla\phi||_2^2 \mathrm{d}\tau) &\le ||\nabla\phi||_{2,2;t}||\phi||_{2,2;t}||\overline{w}||_{\infty,\infty;t} \\ &\le ||\nabla\phi||_{2,2;t}||\phi||_{2,\infty;t}\sqrt{t}||\overline{w}||_{\infty,\infty;t} \\ &\le \frac{1}{2}||\nabla\phi||_{2,2;t}^2\sqrt{t}||\overline{w}||_{\infty,\infty;T} + \frac{1}{2}||\phi||_{2,\infty;t}^2\sqrt{t}||\overline{w}||_{\infty,\infty;T}, \end{aligned} \tag{3.28}$$

for any $0 \le t \le T$.

From the above formula, we can obtain

$$\frac{1}{2}\sup_{0\le\tau\le T'} ||\phi(\tau)||_2^2 + \nu \int_0^{T'} ||\nabla\phi||_2^2 \mathrm{d}\tau \le ||\nabla\phi||_{2,2;T'}^2\sqrt{T'}||\overline{w}||_{\infty,\infty;T} + ||\phi||_{2,\infty;T'}^2\sqrt{T'}||\overline{w}||_{\infty,\infty;T}, \tag{3.29}$$

for any $0 \le T' \le T$.

Let's choose T' in such a way that

$$r := \sqrt{T'}||\overline{w}||_{\infty,\infty;T} \le \min\{\frac{1}{4}, \frac{\nu}{2}\}. \tag{3.30}$$

Then, it follows that

$$(\frac{1}{2} - r)||\phi||_{2,\infty;T'}^2 + (\nu - r)||\nabla\phi||_{2,2;T'}^2 \le 0, \tag{3.31}$$

thus, $||\phi||_{2,\infty;T'} + ||\nabla\phi||_{2,2;T'} = 0$. We can repeat the above procedure if $T' < T$, with ϕ replaced by $\hat{\phi}$ defined by $\hat{\phi}(t) := \phi(t+T'), t \ge 0$. After finitely many steps, we obtain $||\phi||_{2,\infty;T} + ||\nabla\phi||_{2,2;T} = 0$. The proof is complete. □

The following lemma introduce regularity properties of weak solution w of the Navier–Stokes systems (3.11).

Lemma 3.8. *Let $0 < T < \infty, \psi^0 \in W_{0,\sigma}^{1,2}(\Omega), g \in C_0^\infty([0,\infty); C_0^\infty(\Omega)^3)$. And let the function $\overline{u}$ satisfies the bounded condition: $\overline{u} \in L^\infty((0,T)\times\Omega)^3 \cap L^\infty(0,T; L_\sigma^2(\Omega))$. Then, there exists a uniquely determined Leray-Hopf solutions w of the Navier–Stokes systems (3.11), after a redefinition on a null set, $w : [0,T] \to L_\sigma^2(\Omega)$ is strongly continuous, and satisfies*

$$w \in L^\infty(0,T; W_{0,\sigma}^{1,2}(\Omega)) \cap L^2(0,T; W^{2,2}(\Omega)^3), \quad w_t \in L^2(0,T; L_\sigma^2(\Omega)). \tag{3.32}$$

Moreover, suppose additionally that functions $\overline{u}, \overline{w} \in C^{\infty}([0,T]\times\Omega)^3$, then the function w satisfies the following property:

$$w \in C^1((0,T); C^1_\sigma(\Omega)). \tag{3.33}$$

Proof. First of all, the Lemma 3.6 shows that there exists a determined *Leray-Hopf solutions*

$$w \in L^{\infty}(0,T; L^2_\sigma(\Omega)) \cap L^2(0,T; W^{1,2}_{0,\sigma}(\Omega)), \tag{3.34}$$

of the Navier–Stokes systems (3.11) which is weakly continuous, after a redefinition on a null set of in $[0,T]$, and satisfy the integral equation:

$$w(t) = S(t)\psi^0 + (\mathcal{J}Pg)(t) + (\mathcal{J}L(\overline{w})w)(t) + A^{\frac{1}{2}}(\mathcal{J}A^{-\frac{1}{2}}P\mathrm{div}(\overline{u}\otimes w))(t) \tag{3.35}$$

for all $t \in [0,T]$.

Second, since $\overline{u}$ satisfies the bounded condition: $\overline{u} \in L^{\infty}((0,T)\times\Omega)^3 \cap L^{\infty}(0,T; L^2_\sigma(\Omega))$, we can get $\overline{u}(t,x)$ satisfies the following Serrin's condition

$$\begin{aligned} ||\overline{u}||_{q,s;T} = (\int_0^T(\int_\Omega|\overline{u}(t,x)|^4dx)^2dt)^{\frac{1}{8}} &\le ||\overline{u}||^{\frac{1}{2}}_{\infty,\infty;T}(\int_0^T(\int_\Omega|\overline{u}(t,x)|^2dx)^2dt)^{\frac{1}{8}} \\ &\le T^{\frac{1}{8}}||\overline{u}||^{\frac{1}{2}}_{\infty,\infty;T}||\overline{u}||^{\frac{1}{2}}_{2,\infty;T} < +\infty, \end{aligned} \tag{3.36}$$

with $\frac{3}{q}+\frac{2}{s}=1,\ s=8,\ q=4$. By the Lemma 3.7, the *Leray-Hopf solutions* of the Navier–Stokes systems (3.11) is unique.

Third, observe that the function w is also a weak solution of the following linear system

$$w_t - \nu\Delta w + \nabla p = \overline{g}, \quad \mathrm{div} w = 0, \quad w|_{\partial\Omega} = 0, \quad w(0) = \psi^0$$

with $\overline{g} = g + L(\overline{w})w + \overline{u}\cdot\nabla w$. Noting the known conditions: $\overline{u}, \overline{w} \in L^{\infty}((0,T)\times\Omega)^3 \cap L^{\infty}(0,T; L^2_\sigma(\Omega))$, refer to the formulas (4.33) and (4.34), it is easy to prove that the function w satisfy

$$||L(\overline{w})w||_{2,2;T} \le ||\overline{w}||_{\infty,\infty;T}||\nabla w||_{2,2;T}, \quad ||\overline{u}\cdot\nabla w||_{2,2;T} \le ||\overline{u}||_{\infty,\infty;T}||\nabla w||_{2,2;T}. \tag{3.37}$$

Using the Theorem 2.14, it shows that $w : [0,T] \to L^2_\sigma(\Omega)$ is strongly continuous, after a corresponding redefinition. We will apply the basic estimates of the linear theory, applying (2.51) of Theorem 2.15, (2.55), (2.58) of Theorem 2.16, setting

$$|||w|||_T := ||w'||_{2,2;T} + ||Aw||_{2,2;T} + ||A^{\frac{1}{2}}w||_{2,\infty;T} + ||w||_{2,\infty;T}, \tag{3.38}$$

and using

$$\begin{aligned} &||w(t)||_2 \le ||\psi^0||_2 + \int_0^t ||w'(s)||_2 ds, \quad t \in [0,T], \\ &||w||_{2,\infty;T} \le ||\psi^0||_2 + \sqrt{T}||w'||_{2,2;T}, \\ &\nu^{\frac{1}{2}}||\nabla w||_2 = ||A^{\frac{1}{2}}w||_2, \end{aligned} \tag{3.39}$$

it follows that

(3.40)

$$
\begin{aligned}
|||w|||_T &\le C(1+T)(||\psi^0||_2 + ||A^{\frac{1}{2}}\psi^0||_2 + ||g + L(\overline{w})w + \overline{u}\cdot\nabla w||_{2,2;T}) \\
&\le C(1+T)(||\psi^0||_2 + ||A^{\frac{1}{2}}\psi^0||_2 + ||g||_{2,2;T} + ||L(\overline{w})w||_{2,2;T} + ||\overline{u}\cdot\nabla w||_{2,2;T}) \\
&\le C(1+T)(||\psi^0||_2 + ||A^{\frac{1}{2}}\psi^0||_2 + ||g||_{2,2;T} + \nu^{-\frac{1}{2}}\sqrt{T}(||\overline{w}||_{\infty,\infty;T} + ||\overline{u}||_{\infty,\infty;T})||A^{\frac{1}{2}}w||_{2,\infty;T}) \\
&\le C(1+T)(||\psi^0||_2 + ||A^{\frac{1}{2}}\psi^0||_2 + ||g||_{2,2;T} + \nu^{-\frac{1}{2}}\sqrt{T}(||\overline{w}||_{\infty,\infty;T} + ||\overline{u}||_{\infty,\infty;T})|||w|||_T)
\end{aligned}
$$

with $C = C(\Omega,\nu) > 0$ not depending on T.

The following, considering any $(0 <)T'(\le T)$, the estimation formula above also holds if replacing T with T', thus we obtain

(3.41)

$$
|||w|||_{T'} \le C(1+T')(||\psi^0||_2 + ||A^{\frac{1}{2}}\psi^0||_2 + ||g||_{2,2;T'} + \nu^{-\frac{1}{2}}\sqrt{T'}(||\overline{w}||_{\infty,\infty;T'} + ||\overline{u}||_{\infty,\infty;T'})|||w|||_{T'})
$$

with $C = C(\Omega,\nu) > 0$ not depending on T'.

Choosing T' in such a way that

$$
\kappa := C(1+T')\nu^{-\frac{1}{2}}\sqrt{T'}(||\overline{w}||_{\infty,\infty;T} + ||\overline{u}||_{\infty,\infty;T}) < 1, \tag{3.42}
$$

then, it follows that

$$
|||w|||_{T'} \le (1-\kappa)^{-1}C(1+T')(||\psi^0||_2 + ||A^{\frac{1}{2}}\psi^0||_2 + ||g||_{2,2;T'}). \tag{3.43}
$$

Using the Inequality (2.4) of Theorem 2.1 and the interpolation

$$
||\nabla u||_2 \le \nu^{-\frac{1}{2}}(||Au||_2 + ||u||_2), \tag{3.44}
$$

we obtained an inequality about the second derivative

$$
||\nabla^2 u||_2 \le C(||Au||_2 + ||u||_2) \tag{3.45}
$$

with $C = C(\Omega,\nu) > 0$. Combining formula (3.43) above, we have

$$
||w'||_{2,2;T'} + ||\nabla^2 w||_{2,2;T'} + ||\nabla w||_{2,\infty;T'} + ||w||_{2,\infty;T'} < \infty. \tag{3.46}
$$

Since the constant in (3.42) does not dependent on T', we can repeat the above procedure if $T' < T$, with w replaced by $\hat{w}$ defined by $\hat{w}(t) := w(t+T'), t \ge 0$. After finitely many steps, we can obtain the above formula (3.32) for $0 < T < \infty$.

Finally, by formula (3.37), i.e., $L(\overline{w})w,\ \overline{u}\cdot\nabla w \in L^2(0,T;L^2(\Omega)^3)$, and formula (3.32), noting the external force $g \in C_0^\infty([0,\infty);C_0^\infty(\Omega)^3)$ and $\overline{u},\ \overline{w} \in C^\infty([0,T]\times\Omega)^3$, using the same method as ([14], Theorem 1.8.2 of P300), replacing $u\cdot\nabla u$ by $\overline{u}\cdot\nabla w$, replacing f by $g + L(\overline{w})w$, it follows that (3.33) be true. The proof is complete. □

Let's introduce the following analytic lemmas, which are also very interesting.

Lemma 3.9. *Let $h_1, h_2 \in C[a, a+\gamma]$ and $h_1 \in C^1(a, a+\gamma)$, and there exist positive constant Λ, such that*

$$\mu\{t \in (a, a+\gamma); h_1'(t) = h_2(t)\} < \gamma, \quad |h_1(t)| > \Lambda, \quad \textit{for} \quad t \in [a, a+\gamma]. \tag{3.47}$$

Defining the following functions

$$\begin{aligned}
&\widehat{f}(a+\sigma) := \int_a^{a+\sigma} e^{-\int_\tau^{a+\sigma} \frac{h_2(s)}{h_1(s)} ds} \frac{1}{h_1(\tau)}(z(\tau) - \frac{1}{\sigma}\int_a^{a+\sigma} z(r)dr)d\tau, \quad 0 < \sigma \le \gamma, \\
&\widehat{f}(a) := \lim_{\sigma \to 0+} \widehat{f}(a+\sigma) = 0, \\
&z(t) := h_1(t) - \int_a^t h_2(s)ds, \qquad a \le t \le a+\gamma,
\end{aligned} \tag{3.48}$$

then, there exist $0 < \sigma \le \gamma$, such that

$$|\widehat{f}(a+\sigma)| > 0. \tag{3.49}$$

Proof. Suppose that on the contrary, i.e.,

$$\widehat{f}(a+\sigma) \equiv 0, \quad \text{for any} \quad \sigma \in (0, \gamma). \tag{3.50}$$

Then, from the following formula for derivative,

$$\begin{aligned}
\widehat{f}'(a+\sigma) = &\int_a^{a+\sigma} e^{-\int_\tau^{a+\sigma} \frac{h_2(s)}{h_1(s)} ds} \frac{1}{h_1(\tau)}(-\frac{z(a+\sigma)}{\sigma} + \frac{1}{\sigma^2}\int_a^{a+\sigma} z(r)dr)d\tau \\
&+ \frac{1}{h_1(a+\sigma)}(z(a+\sigma) - \frac{1}{\sigma}\int_a^{a+\sigma} z(r)dr) \\
&- \frac{h_2(a+\sigma)}{h_1(a+\sigma)}\widehat{f}(a+\sigma), \qquad a+\sigma \in (a, a+\gamma),
\end{aligned} \tag{3.51}$$

based on the above (3.50), it follows that

$$\begin{aligned}
0 = &\int_a^{a+\sigma} e^{-\int_\tau^{a+\sigma} \frac{h_2(s)}{h_1(s)} ds} \frac{1}{h_1(\tau)}(-\frac{z(a+\sigma)}{\sigma} + \frac{1}{\sigma^2}\int_a^{a+\sigma} z(r)dr)d\tau \\
&+ \frac{1}{h_1(a+\sigma)}(z(a+\sigma) - \frac{1}{\sigma}\int_a^{a+\sigma} z(r)dr), \qquad a+\sigma \in (a, a+\gamma).
\end{aligned} \tag{3.52}$$

This leads to the following

$$0 = (z(a+\sigma) - \frac{1}{\sigma}\int_a^{a+\sigma} z(r)dr)\theta(a+\sigma), \qquad a+\sigma \in (a, a+\gamma), \tag{3.53}$$

where

$$\begin{aligned}
&\theta(a+\sigma) := h_1(a+\sigma)v(a+\sigma) - \sigma, \\
&v(a+\sigma) := \int_a^{a+\sigma} e^{-\int_\tau^{a+\sigma} \frac{h_2(s)}{h_1(s)} ds} \frac{1}{h_1(\tau)} d\tau, \quad \sigma \in (0, \gamma).
\end{aligned} \tag{3.54}$$

Then, it follows that

$$\theta'(a+\sigma) = (h_1'(a+\sigma) - h_2(a+\sigma))v(a+\sigma), \qquad a+\sigma \in (a, a+\gamma), \tag{3.55}$$

and the function v satisfies the following ordinary differential equation:

$$\begin{cases} v'(t) = -\dfrac{h_2(t)}{h_1(t)}v(t) + \dfrac{1}{h_1(t)}, & a \le t \le a+\gamma. \\ v(a) = 0. \end{cases} \tag{3.56}$$

From the known conditions (3.47), there is $a+\sigma_0 \in (a, a+\gamma)$, such that

$$h_1'(a+\sigma_0) - h_2(a+\sigma_0) \neq 0. \tag{3.57}$$

Using local sign preserving property of continuous functions, there is a sufficiently small positive $\delta_1 < \sigma_0$, such that

$$h_1'(a+\sigma) - h_2(a+\sigma) \text{ is not change the sign, for } \sigma_0 - \delta_1 < \sigma < \sigma_0 + \delta_1, \tag{3.58}$$

since $h_1' - h_2 \in C((a, a+\gamma))$.

Besides, obviously, $v(\cdot) \in C[a, a+\gamma]$ and $v(a+\sigma)\big|_{\sigma\in(\sigma_0-\delta_1,\sigma_0+\delta_1)} \equiv 0$ is not the solution of the equation (3.127). Hence, by local sign preserving property of continuous functions, there is $\sigma_1 \in (\sigma_0 - \delta_1, \sigma_0 + \delta_1)$ and $0 < \delta_2 \le \min\{\delta_1 - (\sigma_1 - \sigma_0), \delta_1 + (\sigma_1 - \sigma_0)\}$ such that

$$v(a+\sigma) \text{ is not change the sign, for } \sigma_1 - \delta_2 < \sigma < \sigma_1 + \delta_2. \tag{3.59}$$

Combining (3.55), (3.58) and (3.59), we can easily deduce that the function $\theta(a+\sigma)$ is strictly monotonic, for $\sigma \in (\sigma_1 - \delta_2, \sigma_1 + \delta_2)$.

Using $(a+\sigma_1-\delta_2, a+\sigma_1+\delta_2) \subseteq (a, a+\gamma)$ and $h_1' - h_2 \in C((a, a+\gamma))$, by (3.53) and the conclusion above, we can deduce that

$$z(a+\sigma) - \frac{1}{\sigma}\int_a^{a+\sigma} z(r)dr \equiv 0, \text{ for } \sigma_1 - \delta_2 < \sigma < \sigma_1 + \delta_2. \tag{3.60}$$

Noting $z(a+\sigma) := h_1(a+\sigma) - \int_a^{a+\sigma} h_2(s)ds$ and $(\sigma_1 - \delta_2, \sigma_1 + \delta_2) \subseteq (\sigma_0 - \delta_1, \sigma_0 + \delta_1)$, this is a contradiction between (3.58) and (3.60). The proof is complete. □

Lemma 3.10. *Let $h_1, h_2 \in C[a, a+\gamma], h_1 \in C^1(a, a+\gamma)$ and $h_2(a) \neq 0$, and let function h_1 satisfy one of the following two conditions:*

(1). $h_1(a) = 0$,

(2). there exist positive constant Λ, such that

$$|h_1(t)| > \Lambda, \quad \text{for} \quad t \in [a, a+\gamma]. \tag{3.61}$$

Then, there exist $0 < \sigma \le \gamma$ and a function $g(\cdot) \in W^{1,2}(a, a+\sigma)$, such that

$$\begin{cases} \left|\displaystyle\int_a^{a+\sigma} (g'(t)h_1(t) + g(t)h_2(t))dt\right| \le ||h_2||_2\sqrt{\gamma}, \\ g(a) = 0, \qquad g(a+\sigma) = 1. \end{cases} \tag{3.62}$$

Proof. In order to prove this conclusion we shall discuss it in three cases.

Case (1): $h_1(a) = 0$.

Then, for any $0 < \varepsilon \le \frac{||h_2||_2}{1+||h_2||_2}\sqrt{\gamma}$, there exist a positive $\sigma(\le \varepsilon^2)$, such that $|h_1(t)| < \varepsilon$, for $t \in [a, a+\sigma]$. Let's define the functions below

$$g(t) := cos^2[\frac{\pi}{2\sigma}(t-a-\sigma)], \quad \eta(t) := g'(t)h_1(t) + g(t)h_2(t), \quad a \le t \le a+\sigma. \tag{3.63}$$

Then, it is obviously that

$$g(a) = 0, \qquad g(a+\sigma) = 1. \tag{3.64}$$

It follows that

$$\begin{aligned} |\int_a^{a+\sigma} \eta(t)dt| &= |\int_a^{a+\sigma} (g'(t)h_1(t) + g(t)h_2(t))dt| \\ &\le \int_a^{a+\sigma} |(g'(t)h_1(t)| + |g(t)h_2(t))|dt \\ &\le \varepsilon + ||g||_2||h_2||_2 \le \varepsilon + \sqrt{\sigma}||h_2||_2 \\ &\le (1+||h_2||_2)\varepsilon \le ||h_2||_2\sqrt{\gamma}. \end{aligned} \tag{3.65}$$

Moreover, we have

$$||\eta||_{L^2(a,a+\sigma)} = ||g'h_1 + gh_2||_{L^2(a,a+\sigma)} \le (\frac{2}{\sqrt{\sigma}}\varepsilon + ||h_2||_2) \le (\frac{2}{\sqrt{\sigma}}\sqrt{\gamma}+1)||h_2||_2, \tag{3.66}$$

and

$$||g||_{L^2(a,a+\sigma)} \le \sqrt{\sigma}, \qquad ||g'||_{L^2(a,a+\sigma)} \le \frac{2}{\sqrt{\sigma}}. \tag{3.67}$$

Case (2): there exist positive constant Λ, such that

$$\mu\{t \in (a, a+\gamma); h_1'(t) = h_2(t)\} < \gamma, \qquad |h_1(t)| > \Lambda, \quad \text{for} \quad t \in [a, a+\gamma].$$

Consider the following initial value problem of ordinary differential equation:

$$\begin{cases} g'(t) = -\dfrac{h_2(t)}{h_1(t)}g(t) + \dfrac{\eta(t)}{h_1(t)}, & a \le t \le a+\gamma, \\ g(a) = 0, \end{cases} \tag{3.68}$$

where the function $\eta(\cdot)$ will be given later.

For later applications, we will construct special functions to prove this conclusion. Let's choose a special function $\eta(t)$ which be defined by

$$\eta(t) := \lambda_0 z(t) - \frac{\lambda_0}{\sigma}\int_a^{a+\sigma} z(s)ds, \qquad z(t) := h_1(t) - \int_a^t h_2(s)ds, \quad t \in [a, a+\gamma], \tag{3.69}$$

where two constants $0 < \sigma \leq \gamma$ and λ_0 will be determined below.

Then, it follows that

$$
\begin{aligned}
|\int_a^{a+\sigma}(g'(t)h_1(t)+g(t)h_2(t))dt| &= |\int_a^{a+\sigma}\eta(t)dt| \\
&= |\lambda_0||\int_a^{a+\sigma}(z(s)-\frac{1}{\sigma}\int_a^{a+\sigma}z(\tau)d\tau)ds| = 0.
\end{aligned} \tag{3.70}
$$

From the equation above, we can obtain

$$
g(t) = \int_a^t e^{-\int_\tau^t \frac{h_2(s)}{h_1(s)}ds}\frac{\eta(\tau)}{h_1(\tau)}d\tau, \quad t \in [a, a+\gamma]. \tag{3.71}
$$

Let $g(a+\sigma) = 1$, from (3.71), it follows that $1 = \lambda_0 \widehat{f}(a+\sigma)$, where the function

$$
\begin{aligned}
&\widehat{f}(a+\sigma) := \int_a^{a+\sigma} e^{-\int_\tau^{a+\sigma}\frac{h_2(s)}{h_1(s)}ds}\frac{1}{h_1(\tau)}(z(\tau)-\frac{1}{\sigma}\int_a^{a+\sigma}z(r)dr)d\tau, \quad 0<\sigma\leq\gamma, \\
&\widehat{f}(a) := \lim_{\sigma\to 0+}\widehat{f}(a+\sigma) = 0.
\end{aligned} \tag{3.72}
$$

Noting $\widehat{f}(\cdot) \in C[a, a+\gamma]$ and $\widehat{f}(a) = 0$, let's define two constants

$$
A := \max_{\tau\in[0,\gamma]} |\widehat{f}(a+\tau)|, \text{ and } \quad \sigma := \underset{\tau\in[0,\gamma]}{argmax}\, |\widehat{f}(a+\tau)|, \tag{3.73}
$$

if the function $|\widehat{f}(a+\tau)|$ has multiple maximum points in $[0,\gamma]$, σ will take the largest one. By Lemma 3.9, it is evident to see that $0 < A < +\infty$ and $\sigma > 0$. Thus, there exists a constant $|\lambda_0| = \frac{1}{A} < +\infty$, satisfy formula below

$$
\lambda_0 \widehat{f}(a+\sigma) = 1. \tag{3.74}
$$

That is to say, there exists $\lambda_0 \in \mathbb{R}$, such that $g(a+\sigma) = \lambda_0\widehat{f}(a+\sigma) = 1$. Moreover, noting $h_1, h_2 \in L^2(a, a+\gamma)$, and $|h_1(t)| > \Lambda$, for $t \in [a, a+\gamma]$, we have

$$
||\eta||_{L^2(a,a+\sigma)} = ||g'h_1 + gh_2||_{L^2(a,a+\sigma)} \leq |\lambda_0|(||h_1||_2 + \sigma||h_2||_2), \tag{3.75}
$$

and

$$
\begin{aligned}
||g||_{L^2(a,a+\sigma)} &\leq |\lambda_0|(||h_1||_2+\sigma||h_2||_2)\frac{\sigma}{\Lambda}e^{\frac{\sqrt{\sigma}}{\Lambda}||h_2||_2}, \\
||g'||_{L^2(a,a+\sigma)} &\leq \max_{t\in[[a,a+\sigma]}\{|g(t)|\}\frac{||h_2||_2}{\Lambda} + \frac{||\eta||_{L^2(a,a+\sigma)}}{\Lambda} \\
&\leq \frac{|\lambda_0|(||h_1||_2+\sigma||h_2||_2)}{\Lambda}(e^{\frac{\sqrt{\sigma}}{\Lambda}||h_2||_2}\sqrt{\sigma}||h_2||_2+1).
\end{aligned} \tag{3.76}
$$

Case (3): there exist positive constant Λ, such that

$$
\mu\{t \in (a, a+\gamma); h_1'(t) = h_2(t)\} = \gamma, \quad |h_1(t)| > \Lambda, \quad \text{for} \quad t \in [a, a+\gamma].
$$

Consider the following functions:

$$(3.77)\qquad \begin{cases} g(t) = \int_a^t \frac{\theta(s)}{h_1(s)} ds, \qquad a \le t \le a+\sigma, \\ \theta(s) := \lambda_1 h_1(s) - \frac{\lambda_1}{\sigma} \int_a^{a+\sigma} h_1(\tau) d\tau, \end{cases}$$

where two constants $0 < \sigma \le \gamma$ and λ_1 will be determined below.

Then, it follows that

$$(3.78)\quad \int_a^{a+\sigma} g'(t) h_1(t) dt = \int_a^{a+\sigma} \theta(t) dt = \lambda_1 \int_a^{a+\sigma} (h_1(s) - \frac{1}{\sigma} \int_a^{a+\sigma} h_1(\tau) d\tau) ds = 0, \quad 0 < \sigma \le \gamma.$$

Defining the following functions

$$(3.79)\qquad \begin{aligned} h(a+\sigma) &:= \sigma - \frac{1}{\sigma} \int_a^{a+\sigma} h_1(r) dr \int_a^{a+\sigma} \frac{1}{h_1(\tau)} d\tau, \quad 0 < \sigma \le \gamma, \\ h(a) &:= \lim_{\sigma \to 0+} h(a+\sigma) = 0, \end{aligned}$$

Then, using Cauchy's inequality, noting $\mu\{t \in (a, a+\gamma); h_1'(t) = h_2(t)\} = \gamma$ and $h_2(a) \neq 0$, it follows that there exist $0 < \sigma \le \gamma$, such that

$$(3.80)\qquad |h(a+\sigma)| > 0.$$

Noting function $h(\cdot) \in C[a, a+\gamma]$ and $h(a) = 0$, let's define two constants

$$(3.81)\qquad B := \max_{\tau \in [0,\gamma]} |h(a+\tau)|, \text{ and } \quad \sigma := \mathop{argmax}_{\tau \in [0,\gamma]} |h(a+\tau)|,$$

if the function $|h(a+\tau)|$ has multiple maximum points in $[0, \gamma]$, σ will take the largest one. Then, by the formula (3.80), it is evident to see that $0 < B < +\infty$ and $\sigma > 0$. Thus, there exists a constant $|\lambda_1| = \frac{1}{B} < +\infty$, satisfy formula below

$$(3.82)\qquad \lambda_1 h(a+\sigma) = 1.$$

That is to say, there exists $\lambda_1 \in \mathbb{R}$, such that $g(a+\sigma) = \lambda_1 h(a+\sigma) = 1$. Besides, we have $||g||_{L^\infty(a,a+\sigma)} = |\lambda_1 h(a+\sigma)| = 1$. From formula (3.78), it follows that

$$(3.83)\qquad |\int_a^{a+\sigma} (g'(t) h_1(t) + g(t) h_2(t)) dt| = |\int_a^{a+\sigma} g(t) h_2(t) dt| \le ||h_2||_2 \sqrt{\sigma} \le ||h_2||_2 \sqrt{\gamma}.$$

Moreover, noting $h_1 \in L^2(a, a+\gamma)$, and $|h_1(t)| > \Lambda$, for $t \in [a, a+\gamma]$, we have

$$(3.84)\qquad ||\theta||_{L^2(a,a+\sigma)} = ||g' h_1||_{L^2(a,a+\sigma)} \le |\lambda_1| ||h_1||_2,$$

and

$$(3.85)\qquad \begin{aligned} ||g||_{L^2(a,a+\sigma)} &\le ||g||_{L^\infty(a,a+\sigma)} \sqrt{\sigma} = \sqrt{\sigma}, \\ ||g'||_{L^2(a,a+\sigma)} &\le \frac{||\eta||_{L^2(a,a+\sigma)}}{\Lambda} \le \frac{|\lambda_1|}{\Lambda} ||h_1||_2. \end{aligned}$$

The proof is completed. □

Now, we are going to prove the Theorem 3.1 below. In the process of proving Theorem 3.1, the proof of the result: $||\widetilde{w}-\widetilde{u}||_{L^2(0,T;L^2_\sigma(\Omega))}=0$ is lengthy and technical. Before getting into a detailed proof, let's first briefly look at the idea behind it, which be clarified in the following Remark 3.11.

Remark 3.11. *Let functions $\widetilde{u}(t,x),\widetilde{w}(t,x)$ are two Leray-Hopf solutions of the NSE (1.1). First of all, there is a closed set E of Lebesgue measure zero in $(0,T)$, such that $\widetilde{u},\widetilde{w}$ are smooth in $((0,T)-E)\times\Omega$. Suppose $||\widetilde{w}-\widetilde{u}||_{L^2(0,T;L^2_\sigma(\Omega))}\neq 0$. Then, there exists a small time interval $(\xi,b_{j_0})\subset(0,T)-E$, such that*

$$||\widetilde{w}-\widetilde{u}||_{L^2(0,\xi;L^2_\sigma(\Omega))}=0,\quad \text{and}\quad 4\varepsilon_0:=||\widetilde{w}-\widetilde{u}||_{L^2(\xi,b_{j_0};L^2_\sigma(\Omega))}\neq 0. \tag{3.86}$$

Then, there exists a function $f_0\in C_0^\infty([\xi,b_{j_0});C^\infty_{0,\sigma}(\Omega))$ with $||f_0||_{L^2(\xi,b_{j_0};L^2_\sigma(\Omega))}\leq 2$, such that

$$|\langle\widetilde{w}-\widetilde{u},f_0\rangle_{\Omega,(\xi,b_{j_0})}|\geq||\widetilde{w}-\widetilde{u}||_{L^2(\xi,b_{j_0};L^2_\sigma(\Omega))}. \tag{3.87}$$

On the one hand, by the absolute continuity of integral, for any $(c,d)(\subset(\xi,b_{j_0}))$ that is sufficiently close to (ξ,b_{j_0}), it follows that

$$|\langle\widetilde{w}-\widetilde{u},f_0\rangle_{\Omega,(c,d)}|>2\varepsilon_0. \tag{3.88}$$

On the other hand, from the definition of weak solutions to the NSE (1.1), we can obtain the test formula (3.119) (where the definition of test function space $\mathcal{H}_0$ see (3.107)). We introduce the following backward incompressible equation (3.120) on $[a,d]$ (here $\xi<a<c<d<b_{j_0}$), which admits a weak solution $\psi\in L^\infty(a,d;L^2_\sigma(\Omega))\cap L^2(a,d;W^{1,2}_{0,\sigma}(\Omega))$ satisfy the following regularity conclusion:

$$\psi\in L^\infty(a,d;W^{1,2}_{0,\sigma}(\Omega))\cap L^2(a,d;W^{2,2}(\Omega)^3)\cap C^1((a,d);C^1_\sigma(\Omega)),\quad \psi_t\in L^2(a,d;L^2_\sigma(\Omega)). \tag{3.89}$$

We do not know whether the weak solution ψ can be extended to $[\xi,d]$ such that $\psi\in\mathcal{H}_0$ and still satisfies equation (3.120), since ξ may belong to the set E. This brings essential difficulty to our proof. We expect essentially choose a piecewise continuous function $\widetilde{g}$ (refer to (3.132)), so that $\widetilde{g}\equiv 0$ on (ξ,a), $\widetilde{g}=g$ on $(a,a+\sigma)$, and $\widetilde{g}\equiv 1$ on the set $(a+\sigma,d)$, (where $a+\sigma\leq c$). Then, taking $\varphi(t,x):=\widetilde{g}(t)\psi(t,x)\in\mathcal{H}_0$ as a test function in formula (3.119), it follows that

$$\begin{aligned}
|\langle\widetilde{w}-\widetilde{u},f_0\rangle_{\Omega,(c,d)}|&=|\langle\widetilde{w}-\widetilde{u},\psi_t-A\psi+P\widetilde{u}\cdot\nabla\psi+L(\widetilde{w})\psi\rangle_{\Omega,(c,d)}|\\
&=|\langle\widetilde{w}-\widetilde{u},\varphi_t-A\varphi+P\widetilde{u}\cdot\nabla\varphi+L(\widetilde{w})\varphi\rangle_{\Omega,(c,d)}|\\
&=|-\langle\widetilde{w}-\widetilde{u},\varphi_t-A\varphi+P\widetilde{u}\cdot\nabla\varphi+L(\widetilde{w})\varphi\rangle_{\Omega,(\xi,c)}|\\
&=|\int_a^{a+\sigma}(\widetilde{g}'(t)h_1(t)+\widetilde{g}(t)h_2(t))dt+\int_{a+\sigma}^c(\widetilde{g}'(t)h_1(t)+\widetilde{g}(t)h_2(t))dt|\\
&=|\int_a^{a+\sigma}(g'(t)h_1(t)+g(t)h_2(t))dt+\int_{a+\sigma}^c h_2(t)dt|,
\end{aligned} \tag{3.90}$$

where the functions $h_1(\cdot),h_2(\cdot)$ defined by:

$$h_1(t):=\langle\widetilde{w}(t)-\widetilde{u}(t),\psi(t)\rangle,\quad h_2(t):=\langle\widetilde{w}(t)-\widetilde{u}(t)),f_0(t)\rangle,\quad t\in[a,d]. \tag{3.91}$$

Now, if we could make the right-hand side term of the above arbitrarily small, for example, if it smaller than ε_0, then we will obtain a contradiction with (3.88). We now carry out the details.

Proof. It is obviously that we only need to prove the uniqueness of the *Leray-Hopf solutions* of the NSE (1.1), and other conclusions already hold (see Theorem 2.4 and Theorem 2.5). Let functions $\widetilde{u}(t,x)$ and $\widetilde{w}(t,x)$ are two *Leray-Hopf solutions* of the NSE (1.1). Then, since $\widetilde{w}(t,x)$ be one *Leray-Hopf solutions* of the NSE (1.1), we have

$$E_T(\widetilde{w}) \le 2\|u^0\|_2^2 + 8||f||_{2,1;T}^2, \quad ||\widetilde{w}(t)||_2 \le 2\|u^0\|_2 + 4||f||_{2,1;T}, \tag{3.92}$$

for $0 \le t \le T < +\infty$. Besides, by Lemma 3.3, there is a closed set $E_{\widetilde{w}}$ of Lebesgue measure zero in $(0,T)$, such that $\widetilde{w}$ is smooth in $((0,T) - E_{\widetilde{w}}) \times \Omega$. Let $E_{\widetilde{u}}$ be a closed sets defined in (3.6), and define $E := E_{\widetilde{u}} \cup E_{\widetilde{w}}$, then E is a closed set of Lebesgue measure zero in $(0,T)$, and the functions $\widetilde{u}, \widetilde{w}$ are smooth in $((0,T) - E) \times \Omega$. In addition, by the inequality:

$$\begin{aligned} \big|||(\widetilde{w} - \widetilde{u})(t)||_2 - ||(\widetilde{w} - \widetilde{u})(t_0)||_2\big| &\le ||(\widetilde{w} - \widetilde{u})(t) - (\widetilde{w} - \widetilde{u})(t_0)||_2 \\ &\le ||\widetilde{w}(t) - \widetilde{w}(t_0)||_2 + ||\widetilde{u}(t) - \widetilde{u}(t_0)||_2, \end{aligned}$$

we can deduce that

$$||(\widetilde{w} - \widetilde{u})(t)||_2 \in C((0,T) - E), \tag{3.93}$$

since the Lemma 3.4 leads to the functions $\widetilde{u}, \widetilde{w} : (0,T) - E \to L^2_\sigma(\Omega)$ are strongly continuous.

Let $(a_i, b_i), i \in I$, be the connected components of $(0,T) - E$. Then, obviously $a_1 = 0, a_{i+1} = b_i$, for $(1 \le) i \in I$. We shall prove $\widetilde{w} = \widetilde{u}$ in $[0,T]$, i.e., $||\widetilde{w} - \widetilde{u}||_{2,\infty;T} + ||\nabla\widetilde{w} - \nabla\widetilde{u}||_{2,2;T} = 0$, by reduction to absurdity method.

First of all, suppose $||\widetilde{w} - \widetilde{u}||_{L^2(0,T;L^2_\sigma(\Omega))} \neq 0$, and define the following

$$\xi := \max_\lambda\{0 \le \lambda \le T : ||\widetilde{w} - \widetilde{u}||_{L^2(0,\lambda;L^2_\sigma(\Omega))} = 0\}. \tag{3.94}$$

Then we may assume that $a_{j_0} \le \xi < b_{j_0}, j_0 \in I$, and it follows that

$$||\widetilde{w} - \widetilde{u}||_{L^2(0,\xi;L^2_\sigma(\Omega))} = 0, \quad \text{and} \quad 4\varepsilon_0 := ||\widetilde{w} - \widetilde{u}||_{L^2(\xi,b_{j_0};L^2_\sigma(\Omega))} \neq 0. \tag{3.95}$$

Noting the formulas $||\widetilde{u}||_{2,2;T} \le \sqrt{T}(2||u^0||_2 + 4||f||_{2,1;T}), ||\widetilde{w}||_{2,2;T} \le \sqrt{T}(2||u^0||_2 + 4||f||_{2,1;T})$, by the absolute continuity of integral, there exists a sufficiently small positive $\delta_1 (< \frac{b_{j_0} - \xi}{8})$, such that

$$|\int_e \langle \widetilde{w} - \widetilde{u}, \widetilde{w} - \widetilde{u}\rangle \, dt| < \frac{\varepsilon_0^2}{16}, \tag{3.96}$$

for any Lebesgue's measurable subset e of $[0,T]$ with $\mu(e) \le \delta_1$.

Since $C_0^\infty([\xi, b_{j_0}); C^\infty_{0,\sigma}(\Omega))$ is density in $L^2(\xi, b_{j_0}; L^2_\sigma(\Omega))$, by the Hahn-Banach theorem, there exists a smooth function

$$f_0 \in C_0^\infty([\xi, b_{j_0}); C^\infty_{0,\sigma}(\Omega)),$$

with $||f_0||_{L^2(\xi,b_{j_0};L^2_\sigma(\Omega))} \le 2$, such that the following two conclusions hold:

$$\begin{aligned}&(1).\quad |\langle \widetilde{w}-\widetilde{u}, f_0\rangle_{\Omega,(\xi,b_{j_0})}| \ge ||\widetilde{w}-\widetilde{u}||_{L^2(\xi,b_{j_0};L^2_\sigma(\Omega))},\\ &(2).\quad \underset{t\in[\xi,b_{j_0}]}{essinf}\{t; \langle \widetilde{w}(t)-\widetilde{u}(t), f_0(t)\rangle = 0\} = \xi \quad \text{or} \quad \langle \widetilde{w}(\xi)-\widetilde{u}(\xi), f_0(\xi)\rangle \ne 0.\end{aligned} \tag{3.97}$$

Where the conclusion (2), roughly speaking, ξ is the essential zero of function $\langle \widetilde{w}(\cdot)-\widetilde{u}(\cdot), f_0(\cdot)\rangle$ or satisfy $\langle \widetilde{w}(\xi)-\widetilde{u}(\xi), f_0(\xi)\rangle \ne 0$.

By Theorem 2.3, it follows that the functions

$$\widetilde{u}, \widetilde{w} : [0, T) \to L^2_\sigma(\Omega) \text{ are weakly continuous.} \tag{3.98}$$

Combining formulas (3.4), (3.92) and (3.98), noting the function $f_0 : [\xi, b_{j_0}] \to L^2_\sigma(\Omega)$ be strongly continuous since $||f_0(t) - f_0(t_0)||_{L^2_\sigma(\Omega)} \le \int_{t_0}^t ||\frac{\partial f_0}{\partial \tau}||_{L^2_\sigma(\Omega)} d\tau$ for $\xi \le t_0 \le t \le b_{j_0}$, according to the following inequality estimates

$$\begin{aligned}&\big|\langle \widetilde{w}(t)-\widetilde{u}(t), f_0(t)\rangle - \langle \widetilde{w}(t_0)-\widetilde{u}(t_0), f_0(t_0)\rangle\big|\\ &\quad\le \big|\langle \widetilde{w}(t)-\widetilde{u}(t), f_0(t)\rangle - \langle \widetilde{w}(t)-\widetilde{u}(t), f_0(t_0)\rangle\big| + \big|\langle \widetilde{w}(t)-\widetilde{u}(t), f_0(t_0)\rangle - \langle \widetilde{w}(t_0)-\widetilde{u}(t_0), f_0(t_0)\rangle\big|\\ &\quad\le \big|\langle \widetilde{w}(t)-\widetilde{u}(t), f_0(t)-f_0(t_0)\rangle\big| + \big|\langle \widetilde{w}(t)-\widetilde{w}(t_0), f_0(t_0)\rangle\big| + \big|\langle \widetilde{u}(t)-\widetilde{u}(t_0), f_0(t_0)\rangle\big|\\ &\quad\le (4\|u^0\|_2 + 8||f||_{2,1;T})||f_0(t)-f_0(t_0)||_{L^2_\sigma(\Omega)} + \big|\langle \widetilde{w}(t)-\widetilde{w}(t_0), f_0(t_0)\rangle\big| + \big|\langle \widetilde{u}(t)-\widetilde{u}(t_0), f_0(t_0)\rangle\big|,\end{aligned}$$

we can deduce that the unary function $\langle \widetilde{w}(\cdot)-\widetilde{u}(\cdot), f_0(\cdot)\rangle$ is continuous on $[\xi, b_{j_0}]$. Thus, we can choose a sufficiently small positive $\delta(\le \delta_1)$, such that

$$\begin{aligned}&16(\|u^0\|_2 + 2||f||_{2,1;T})\sqrt{\delta} \le \varepsilon_0,\\ &\text{and } \left\langle \widetilde{w}(\xi+\frac{\delta}{2}) - \widetilde{u}(\xi+\frac{\delta}{2}), f_0(\xi+\frac{\delta}{2})\right\rangle \ne 0.\end{aligned} \tag{3.99}$$

For the sake of simplicity, we will denote

$$a := \xi + \frac{\delta}{2}, \qquad c := \xi + \delta, \qquad d := b_{j_0} - \delta.$$

Applying the two formulas above (3.95) and (3.96), we can deduce that

$$||\widetilde{w}-\widetilde{u}||_{L^2(c,d;L^2_\sigma(\Omega))} > 3\varepsilon_0. \tag{3.100}$$

From formulas (3.96), (3.97) and (3.100), we can deduce that

$$\begin{aligned}|\langle \widetilde{w}-\widetilde{u}, f_0\rangle_{\Omega,(c,d)}| &\ge |\langle \widetilde{w}-\widetilde{u}, f_0\rangle_{\Omega,(\xi,d)}| - |\langle \widetilde{w}-\widetilde{u}, f_0\rangle_{\Omega,(\xi,c)}|\\ &\ge ||\widetilde{w}-\widetilde{u}||_{L^2(\xi,d;L^2_\sigma(\Omega))} - |\langle \widetilde{w}-\widetilde{u}, f_0\rangle_{\Omega,(\xi,c)}| > 2\varepsilon_0.\end{aligned} \tag{3.101}$$

Besides, by Lemma 3.3, we can obtain $\widetilde{u}, \widetilde{w} \in C^\infty([a,d]\times\Omega)^3$, combining $\widetilde{u}\big|_{\partial\Omega} = 0, \widetilde{w}\big|_{\partial\Omega} = 0$, (3.4) and (3.92), which leads to the functions $\widetilde{u}, \widetilde{w}$ satisfy the below conditions:

$$\widetilde{u}, \widetilde{w} \in L^\infty((a,d)\times\Omega)^3 \cap L^\infty(a,d;L^2_\sigma(\Omega)) \cap C^\infty([a,d]\times\Omega)^3. \tag{3.102}$$

Since $\widetilde{u}, \widetilde{w}$ are weak solutions of the Navier-Stokes system (1.1) with data f, u^0, we can obtain

$$\begin{aligned} &-\langle \widetilde{u}, \phi_t\rangle_{\Omega,T} - \nu \langle \widetilde{u}, \triangle\phi\rangle_{\Omega,T} + \langle \widetilde{u}\cdot\nabla\widetilde{u}, \phi\rangle_{\Omega,T} = \left\langle u^0, \phi(0)\right\rangle_{\Omega} + \langle f, \phi\rangle_{\Omega,T}\,, \\ &-\langle \widetilde{w}, \phi_t\rangle_{\Omega,T} - \nu \langle \widetilde{w}, \triangle\phi\rangle_{\Omega,T} + \langle \widetilde{w}\cdot\nabla\widetilde{w}, \phi\rangle_{\Omega,T} = \left\langle u^0, \phi(0)\right\rangle_{\Omega} + \langle f, \phi\rangle_{\Omega,T}\,, \end{aligned} \tag{3.103}$$

for any $\phi \in C_0^\infty([0,T); C_{0,\sigma}^\infty(\Omega))$. Combining two formulas above, using integration by parts and the formula div $\widetilde{u} = 0$, it follows that

$$\langle \widetilde{w}-\widetilde{u}, \phi_t - A\phi + P\widetilde{u}\cdot\nabla\phi + L(\widetilde{w})\phi\rangle_{\Omega,(0,T)} = 0, \quad \text{for any} \quad \phi \in C_0^\infty([0,T); C_{0,\sigma}^\infty(\Omega)), \tag{3.104}$$

where $A = -\nu P\Delta$ is the Stokes operator, P is a Helmholtz projection operator, $L(\widetilde{w}) : W_{0,\sigma}^{1,2}(\Omega) \to L_\sigma^2(\Omega)$ is a linear operator satisfying

$$\langle L(\widetilde{w})w, u\rangle := \langle \nabla w, u\otimes\widetilde{w}\rangle\,, \ \text{for} \ \ u \in L_\sigma^2(\Omega), \ 0 < t \le T < +\infty. \tag{3.105}$$

In addition, the formula (3.104) also holds if replacing T with d, from this formula and formula (3.95), we can obtain

$$\langle \widetilde{w}-\widetilde{u}, \phi_t - A\phi + P\widetilde{u}\cdot\nabla\phi + L(\widetilde{w})\phi\rangle_{\Omega,(\xi,d)} = 0, \quad \text{for any} \quad \phi \in C_0^\infty([\xi,d); C_{0,\sigma}^\infty(\Omega)). \tag{3.106}$$

The following, we will prove that (3.134) holds, Let's first prove the formula (3.119).

Noting $C_0^\infty([\xi,d); C_{0,\sigma}^\infty(\Omega))$ be density in the set:

$$\mathcal{H}_0 := \{u \in L^\infty(\xi,d; W_{0,\sigma}^{1,2}(\Omega)) \cap L^2(\xi,d; W^{2,2}(\Omega)^3);\ u_t \in L^2(\xi,d; L_\sigma^2(\Omega)),\ u(d) = 0\}, \tag{3.107}$$

with norm

$$|||u||| := ||A^{\frac{1}{2}}u||_{2,2;(\xi,d)} + ||u||_{2,2;(\xi,d)} + ||u_t||_{2,2;(\xi,d)}, \quad A := -\nu P\Delta, \tag{3.108}$$

then, for any $\varphi \in \mathcal{H}_0$, there is sequence $(\phi^{(k)})_{k=1}^\infty \in C_0^\infty([\xi,d); C_{0,\sigma}^\infty(\Omega))$ with $||A^{\frac{1}{2}}\phi^{(k)}||_{2,\infty;(\xi,d)} \le 2||A^{\frac{1}{2}}\varphi||_{2,\infty;(\xi,d)} + 1$ for $k \in \mathbb{N}$, such that

$$\lim_{k\to\infty} |||\phi^{(k)} - \varphi||| = 0. \tag{3.109}$$

Based on the above conditions, there is $k_1 \in \mathbb{N}$, such that

$$||A^{\frac{1}{2}}(\phi^{(k)} - \varphi)||_{2,\infty;(\xi,d)} \le Q_\varphi := 3||A^{\frac{1}{2}}\varphi||_{2,\infty;(\xi,d)} + 1, \ \text{for} \ k > k_1. \tag{3.110}$$

Using the following relationship

$$\left\langle \widetilde{w}-\widetilde{u}, P\widetilde{u}\cdot\nabla(\phi^{(k)}-\varphi)\right\rangle_{\Omega} = -\left\langle \nabla(\widetilde{w}-\widetilde{u}), \widetilde{u}\otimes(\phi^{(k)}-\varphi)\right\rangle_{\Omega}, \tag{3.111}$$

noting $\widetilde{w}, \widetilde{u} \in L^\infty(0,T; L_\sigma^2(\Omega)) \cap L^2(0,T; W_{0,\sigma}^{1,2}(\Omega))$, formula $\nu^{\frac{1}{2}}||\nabla u||_2 = ||A^{\frac{1}{2}}u||_2$, and b) of Lemma 2.11, combining formulas (3.4), (3.110) and the formula (refer to (3.24) and Lemma 2.10 with

$q=4, \alpha=\frac{3}{8}, s'=\frac{8}{3}$, the interpolation inequality: $||(A^{\frac{1}{2}})^{2\alpha}u||_2 \leq ||A^{\frac{1}{2}}u||_2^{2\alpha}||u||_2^{1-2\alpha}$ with $\alpha=\frac{3}{8}$):

$$
\begin{aligned}
||\widetilde{u}\otimes(\phi^{(k)}-\varphi)||_{L^2(\xi,d;L^2_\sigma(\Omega))} &\leq C||\widetilde{u}||_{L^{\frac{8}{3}}(\xi,d;L^4(\Omega)^3)}||\phi^{(k)}-\varphi||_{L^8(\xi,d;L^4(\Omega)^3)} \\
&\leq C\nu^{-\frac{3}{8}}||\widetilde{u}||_{L^{\frac{8}{3}}((\xi,d;L^4(\Omega)^3)}||A^{\frac{3}{8}}(\phi^{(k)}-\varphi)||_{L^8(\xi,d;L^2(\Omega)^3)} \\
&\leq C'E_T(\widetilde{u})^{\frac{1}{2}}||A^{\frac{1}{2}}(\phi^{(k)}-\varphi)||^{\frac{3}{4}}_{2,\infty;(\xi,d)}||\phi^{(k)}-\varphi||^{\frac{1}{4}}_{2,2;(\xi,d)} \\
&\leq C'(\sqrt{2}\|u^0\|_2+2\sqrt{2}||f||_{2,1;T})Q_\varphi^{\frac{3}{4}}||\phi^{(k)}-\varphi||^{\frac{1}{4}}_{2,2;(\xi,d)},
\end{aligned}
\tag{3.112}
$$

it follows that

$$
\lim_{k\to\infty}\left\langle \widetilde{w}-\widetilde{u}, P\widetilde{u}\cdot\nabla(\phi^{(k)}-\varphi)\right\rangle_{\Omega,(\xi,d)}=0. \tag{3.113}
$$

In addition, by (3.109), it is clear that

$$
\lim_{k\to\infty}\left|\left\langle \widetilde{w}-\widetilde{u}, \frac{\partial(\phi^{(k)}-\varphi)(t)}{\partial t}\right\rangle_{\Omega,(\xi,d)}\right| \leq \lim_{k\to\infty}||\widetilde{w}-\widetilde{u}||_{2,2;T}||\frac{\partial(\phi^{(k)}-\varphi)}{\partial t}||_{L^2(\xi,d;L^2_\sigma(\Omega))}=0. \tag{3.114}
$$

Besides, applying Lemma 2.10, we choose $\alpha=\frac{n}{8}$, $q=4, 2\alpha+\frac{n}{4}=\frac{n}{2}, n=3$, using Holder inequality and the interpolation inequality $||A^\beta u||_2 \leq ||Au||_2^\beta||u||_2^{1-\beta}, u\in D(A),\quad 0\leq\beta\leq 1$, we can obtain

$$
\begin{aligned}
||u\otimes w||_2 &\leq C||u||_4||w||_4 \\
&\leq C'||A^\alpha u||_2||A^\alpha w||_2 \\
&\leq C'||A^{\frac{1}{2}}u||_2^{\frac{n}{4}}||u||_2^{1-\frac{n}{4}}||A^{\frac{1}{2}}w||_2^{\frac{n}{4}}||w||_2^{1-\frac{n}{4}}
\end{aligned}
\tag{3.115}
$$

and this leads to

$$
||u\otimes w||_{2,4/n;T} \leq C||A^{\frac{1}{2}}u||^{n/4}_{2,2;T}||u||^{1-\frac{n}{4}}_{2,\infty;T}||A^{\frac{1}{2}}w||^{n/4}_{2,2;T}||w||^{1-\frac{n}{4}}_{2,\infty;T} \leq C\sqrt{E_T(u)E_T(w)}. \tag{3.116}
$$

Combining the above formula and formulas (3.4), (3.92), (3.105), (3.110), we can deduce

$$
\begin{aligned}
&\lim_{k\to\infty}\left|\left\langle \widetilde{w}-\widetilde{u}, L(\widetilde{w})(\phi^{(k)}-\varphi)\right\rangle_{\Omega,(\xi,d)}\right| = \lim_{k\to\infty}\left|\left\langle \nabla(\phi^{(k)}-\varphi), (\widetilde{w}-\widetilde{u})\otimes\widetilde{w}\right\rangle_{\Omega,(\xi,d)}\right| \\
&\leq \lim_{k\to\infty} C||(\widetilde{w}-\widetilde{u})\otimes\widetilde{w}||_{L^{\frac{4}{3}}(\xi,d;L^2(\Omega)^3)}||\nabla(\phi^{(k)}-\varphi)||_{L^4(\xi,d;L^2(\Omega)^3)} \\
&\leq \lim_{k\to\infty} C'\sqrt{E_T(\widetilde{w}-\widetilde{u})E_T(\widetilde{w})}||A^{\frac{1}{2}}(\phi^{(k)}-\varphi)||^{\frac{1}{2}}_{2,\infty;(\xi,d)}||A^{\frac{1}{2}}(\phi^{(k)}-\varphi)||^{\frac{1}{2}}_{L^2(\xi,d;L^2(\Omega)^3)} \\
&\leq \lim_{k\to\infty} C''(2\|u^0\|_2^2+8||f||^2_{2,1;T})Q_\varphi^{\frac{1}{2}}||A^{\frac{1}{2}}(\phi^{(k)}-\varphi)||^{\frac{1}{2}}_{L^2(\xi,d;L^2(\Omega)^3)}=0,
\end{aligned}
\tag{3.117}
$$

By formulas (3.4) and (3.92), it follows that $(\widetilde{w}-\widetilde{u})\in L^2(0,T;W^{1,2}_{0,\sigma}(\Omega))$. Thus, we also have the following formula

$$
\begin{aligned}
&\lim_{k\to\infty}\left|\left\langle \widetilde{w}-\widetilde{u}, A(\phi^{(k)}-\varphi)\right\rangle_{\Omega,(\xi,d)}\right| \\
&= \lim_{k\to\infty}\left|\left\langle A^{\frac{1}{2}}(\widetilde{w}-\widetilde{u}), A^{\frac{1}{2}}(\phi^{(k)}-\varphi)\right\rangle_{\Omega,(\xi,d)}\right| \\
&\leq \lim_{k\to\infty}(4\|u^0\|_2+8||f||_{2,1;T})||A^{\frac{1}{2}}(\phi^{(k)}-\varphi)||_{L^2(\xi,d;L^2_\sigma(\Omega))}=0.
\end{aligned}
\tag{3.118}
$$

Noting $(\phi^{(k)})_{k=1}^{\infty} \in C_0^{\infty}([\xi, d); C_{0,\sigma}^{\infty}(\Omega))$, basing on formulas (3.104), (3.113), (3.114), (3.117) and (3.118), we have test formula:

$$\langle \widetilde{w} - \widetilde{u}, \varphi_t - A\varphi + P\widetilde{u} \cdot \nabla \varphi + L(\widetilde{w})\varphi \rangle_{\Omega,(\xi,d)} = 0, \text{ for any } \varphi \in \mathcal{H}_0. \tag{3.119}$$

Consider the following linear nonhomogeneous backward incompressible equation:

$$\begin{cases} \psi_t - A\psi + P\widetilde{u} \cdot \nabla \psi + L(\widetilde{w})\psi = Pf_0, & a \le t \le d, x \in \Omega, \\ \psi(d, x) = 0, & x \in \Omega, \\ \psi(t, x) = 0, & a \le t \le d, x \in \partial\Omega, \end{cases} \tag{3.120}$$

where $A = -\nu P \Delta$ is the Stokes operator, P be the Helmholtz projection operator of $L^2(\Omega)^3$ onto $L^2_\sigma(\Omega)$.

Let's make the following time-reverse transformations:

$$\begin{aligned} &w(t, x) := \psi(d - t, x), \quad \overline{u}(t, x) := \widetilde{u}(d - t, x), \quad \overline{w}(t, x) := \widetilde{w}(d - t, x), \\ &\overline{f}_0(t, x) := f_0(d - t, x), \quad (t, x) \in [0, d - a] \times \Omega. \end{aligned} \tag{3.121}$$

Then, the above equation can be reduced to the following linear nonhomogeneous forward incompressible equation:

$$\begin{cases} w_t + Aw - P\overline{u} \cdot \nabla w - L(\overline{w})w = -P\overline{f}_0, & 0 \le t \le d - a, \ x \in \Omega, \\ w(0, x) = 0, & x \in \Omega, \\ w(t, x) = 0, & 0 \le t \le d - a, \ x \in \partial\Omega, \end{cases} \tag{3.122}$$

By Lemma 3.6, replacing T with $d - a$, replacing g with $-\overline{f}_0$, there exists a determined *Leray-Hopf solution*

$$w \in L^\infty(0, d - a; L^2_\sigma(\Omega)) \cap L^2(0, d - a; W^{1,2}_{0,\sigma}(\Omega)), \tag{3.123}$$

of the Navier–Stokes systems (3.122), which satisfy the energy inequality:

$$\frac{1}{2}||w||^2_{2,\infty;d-a} + \nu||\nabla w||^2_{2,2;d-a} \le C_0(\overline{w}, \nu, d - a)||\overline{f}_0||^2_{2,1;d-a}, \tag{3.124}$$

where $C_0(\overline{w}, \nu, d - a)$ be a constant. In addition, noting (3.102) and (3.121), applying Lemma 3.8, we know that, after a redefinition on a null set, the function $w : [0, d - a] \to L^2_\sigma(\Omega)$ is strongly continuous, and satisfy:

(3.125)

$$w \in L^\infty(0, d-a; W^{1,2}_{0,\sigma}(\Omega)) \cap L^2(0, d-a; W^{2,2}(\Omega)^3) \cap C^1((0, d-a); C^1_\sigma(\Omega)), \ \ w_t \in L^2(0, d-a; L^2_\sigma(\Omega)).$$

Therefore, the Navier–Stokes systems (3.120) exists a determined weak solution

$$\psi = \psi(t, x) := w(d - t, x) \in L^\infty(a, d; L^2_\sigma(\Omega)) \cap L^2(a, d; W^{1,2}_{0,\sigma}(\Omega)), \tag{3.126}$$

after a redefinition on a null set, the function $\psi : [a, d] \to L^2_\sigma(\Omega)$ is strongly continuous, which satisfy the following regularity conclusion:

$$\psi \in L^\infty(a, d; W^{1,2}_{0,\sigma}(\Omega)) \cap L^2(a, d; W^{2,2}(\Omega)^3) \cap C^1((a, d); C^1_\sigma(\Omega)), \quad \psi_t \in L^2(a, d; L^2_\sigma(\Omega)), \tag{3.127}$$

and the energy inequality:

$$\frac{1}{2}||\psi||^2_{2,\infty;[a,d]} + \nu||\nabla\psi||^2_{2,2;[a,d]} \le C_0(\widetilde{w}, \nu, d-a)||f_0||^2_{2,1;[a,d]}, \tag{3.128}$$

where $C_0(\overline{w}, \nu, d-a)$ be a constant.

The following, let's define two functions

$$h_1(t) := \langle \widetilde{w}(t) - \widetilde{u}(t), \psi(t) \rangle \,, \quad h_2(t) := \langle \widetilde{w}(t) - \widetilde{u}(t), f_0(t) \rangle \,, \quad t \in [a, d], \tag{3.129}$$

then, by formulas (3.4), (3.92), (3.128), (3.102), (3.127), (3.99), (3.98) and $f_0 \in C^\infty_0([\xi, b_{j_0}); C^\infty_{0,\sigma}(\Omega))$, it is clear that

$$h_1, h_2 \in C[a, d], \quad h_1 \in C^1(a, d) \quad \text{and} \quad h_2(a) \neq 0. \tag{3.130}$$

Now, there are two cases:

Case 1. $h_1(a) = 0$, we choose positive constant $\gamma = \frac{\delta}{2}$.

Case 2. $h_1(a) \neq 0$, by local sign preserving property of continuous functions, there exists a positive constant $\gamma \le \frac{\delta}{2}$, such that $|h_1(t)| > \Lambda := \frac{|h_1(a)|}{2} > 0$, for $t \in [a, a + \gamma]$.

Noting (3.130), applying the Lemma 3.10 for the above two cases, it follows that there exists a positive constant $\sigma \le \gamma$ and a function $g(\cdot) \in W^{1,2}(a, a + \sigma)$, such that

$$\begin{cases} \left| \int_a^{a+\sigma} (g'(t)h_1(t) + g(t)h_2(t))dt \right| \le ||h_2||_{L^2(a,a+\gamma)}\sqrt{\gamma}, \\ g(a) = 0, \qquad g(a+\sigma) = 1. \end{cases} \tag{3.131}$$

Defining the piecewise function:

$$\widetilde{g}(t) := \begin{cases} 1, & a + \sigma \le t \le d, \\ g(t), & a \le t \le a + \sigma, \\ 0, & \xi \le t \le a. \end{cases} \tag{3.132}$$

extending the function $\psi(t, x)$ by zero to region $[\xi, d) \times \Omega$, and defining $\varphi(t, x) := \widetilde{g}(t)\psi(t, x)$, by the formula (3.127), it follows that $\varphi(t, x) \in \mathcal{H}_0$ be a test function (refer to (3.107)). Thus, combining

(3.119), (3.4) and (3.92), using formulas (3.96) and (3.99), applying (3.131), we can deduce that

$$
\begin{aligned}
\big|\, \langle \widetilde{w}-\widetilde{u}, f_0\rangle_{\Omega,(c,d)} \,\big| &= \big|\, \langle \widetilde{w}-\widetilde{u}, \psi_t - A\psi + P\widetilde{u}\cdot\nabla\psi + L(\widetilde{w})\psi\rangle_{\Omega,(c,d)} \,\big| \\
&= \big|\, \langle \widetilde{w}-\widetilde{u}, \varphi_t - A\varphi + P\widetilde{u}\cdot\nabla\varphi + L(\widetilde{w})\varphi\rangle_{\Omega,(c,d)} \,\big| \\
&= \big|\, -\langle \widetilde{w}-\widetilde{u}, \varphi_t - A\varphi + P\widetilde{u}\cdot\nabla\varphi + L(\widetilde{w})\varphi\rangle_{\Omega,(\xi,c)} \,\big| \\
&= \Big| \int_a^{a+\sigma} (\widetilde{g}'(t)h_1(t) + \widetilde{g}(t)h_2(t))dt + \int_{a+\sigma}^{c} (\widetilde{g}'(t)h_1(t) + \widetilde{g}(t)h_2(t))dt \Big| \\
&= \Big| \int_a^{a+\sigma} (g'(t)h_1(t) + g(t)h_2(t))dt + \int_{a+\sigma}^{c} h_2(t)dt \Big| \\
&\le ||h_2||_{L^2(a,a+\gamma)}\sqrt{\gamma} + ||h_2||_{L^2(a,c)}\sqrt{\delta} \le 2||h_2||_{L^2(a,c)}\sqrt{\delta} \\
&\le 2||\widetilde{w}-\widetilde{u}||_{L^\infty(a,c;L^2_\sigma(\Omega))}||f_0||_{L^2(a,c;L^2_\sigma(\Omega))}\sqrt{\delta} \\
&\le 16(\|u^0\|_2 + 2||f||_{2,1;T})\sqrt{\delta} \le \varepsilon_0.
\end{aligned} \tag{3.133}
$$

This is a contradiction between this formula and (3.101). Thus, for any $(0<)T(<\infty)$, we have

$$||\widetilde{w}-\widetilde{u}||_{L^2(0,T;L^2_\sigma(\Omega))} = 0. \tag{3.134}$$

Next, from this formula and formula (3.93) above, noting E is a closed set of Lebesgue measure zero in $(0,T)$, we can deduce

$$||\widetilde{w}-\widetilde{u}||_{2,\infty;T} = 0. \tag{3.135}$$

Finally, since formula $\nu^{\frac{1}{2}}||\nabla w||_2 = ||A^{\frac{1}{2}}w||_2$ and

$$\Big|\Big\langle A^{\frac{1}{2}}(\widetilde{w}-\widetilde{u}), \phi\Big\rangle_{\Omega,(0,T)}\Big| = \Big|\Big\langle \widetilde{w}-\widetilde{u}, A^{\frac{1}{2}}\phi\Big\rangle_{\Omega,(0,T)}\Big| \le ||\widetilde{w}-\widetilde{u}||_{L^2(0,T;L^2_\sigma(\Omega))}||A^{\frac{1}{2}}\phi||_{L^2(0,T;L^2_\sigma(\Omega))} = 0, \tag{3.136}$$

for any $\phi \in C_0^\infty([0,T);C^\infty_{0,\sigma}(\Omega))$, it follows that $||\nabla\widetilde{w}-\nabla\widetilde{u}||_{2,2;T} = 0$.

Based on the above conclusions, we have

$$||\widetilde{w}-\widetilde{u}||_{2,\infty;T} + ||\nabla\widetilde{w}-\nabla\widetilde{u}||_{2,2;T} = 0. \tag{3.137}$$

The proof is complete. □

4 Existence of weak solutions of a particular Navier-Stokes Equations

In this subsection, let $\Omega = \mathbb{R}^3$ or $\Omega \subset \mathbb{R}^3$ be an open nonempty connected bounded uniform C^2-domain with boundary $\partial\Omega$, and Ω is also a C^∞-domain. We will use some arguments from Sohr's book [14], since the Yosida approximation has several advantages, to prove the existence of weak solution w of the Navier-Stokes system (3.11), we consider the weak solutions $w = w^{(k)}$, $k \in \mathbb{N}$ of the following approximate linear systems (4.1), and carry out the limit as $k \to +\infty$ in a certain weak sense.

First we prove some important properties of the following linear nonhomogeneous incompressible Navier-Stokes systems:

$$(4.1)\qquad \begin{cases} w_t + Aw - PJ_k\overline{u}\cdot\nabla w - L(\overline{w})w = Pg, & 0\le t\le T<+\infty, x\in\Omega \\ w(0,x)=\psi^0(x), & x\in\Omega \\ w(t,x)=0, & t\ge 0, x\in\partial\Omega, \end{cases}$$

where $A=-\nu P\Delta$ is the Stokes operator, P is a Helmholtz projection operator, $L(\overline{w}): W^{1,2}_{0,\sigma}(\Omega)\to L^2_\sigma(\Omega)$ is a linear operator satisfying

$$(4.2)\qquad \langle L(\overline{w})w, u\rangle := \langle \nabla w, u\otimes\overline{w}\rangle\,,\ \text{for}\ \ u\in L^2_\sigma(\Omega),\ 0\le t\le T<+\infty,$$

$J_k := (I+k^{-1}A^{\frac12})^{-1}, k\in\mathbb{N}$, is the Yosida approximation operators ([20]), the functions $\overline{u}\in L^\infty(0,T;L^2_\sigma(\Omega))\cap L^2(0,T;W^{1,2}_{0,\sigma}(\Omega))$, $\overline{w}\in L^\infty((0,T)\times\Omega)^3\cap L^\infty(0,T;L^2_\sigma(\Omega))$, the external force g and the initial solenoidal function $\psi^0(x)$ with $\psi^0|_{\partial\Omega}=0$ are given.

Lemma 4.1. *Let $\psi^0\in L^2_\sigma(\Omega)$, $g\in L^1(0,T;L^2(\Omega)^3)$, and let $w=w^{(k)}\in L^\infty(0,T;L^2_\sigma(\Omega))\cap L^2(0,T;W^{1,2}_{0,\sigma}(\Omega))$ be a weak solution of the Navier-Stokes system (4.1) with data g,ψ^0. Then, after a redefinition on a null set of $[0,T), w:[0,T]\to L^2_\sigma(\Omega)$ has the following properties:*

a) $w:[0,T]\to L^2_\sigma(\Omega)$ is strongly continuous with $w(0)=\psi^0$

b) $(\mathcal{J}A^{-\frac12}P\mathrm{div}(J_k\overline{u}\otimes w))(t)\in D(A^{\frac12})$ and

$$(4.3)\qquad w(t)=S(t)\psi^0+(\mathcal{J}Pg)(t)+(\mathcal{J}L(\overline{w})w)(t)+A^{\frac12}(\mathcal{J}A^{-\frac12}P\mathrm{div}(J_k\overline{u}\otimes w))(t)$$

for all $t\in[0,T]$.

c)

$$(4.4)\qquad \frac12||w(t)||_2^2+\nu\int_0^t||\nabla w||_2^2\mathrm{d}\tau=\frac12||\psi^0||_2^2+\int_0^t\langle g,w\rangle\,d\tau+\int_0^t\langle\nabla w,w\otimes\overline{w}\rangle\,\mathrm{d}\tau$$

for all $t\in[0,T]$.

d)

$$(4.5)\qquad \frac12||w||^2_{2,\infty;T}+\nu||\nabla w||^2_{2,2;T}\le C_0(\overline{w},\nu,T)||\psi^0||_2^2+C_0(\overline{w},\nu,T)||g||^2_{2,1;T}$$

for all T, where $C_0(\overline{w},\nu,T)$ be a constant.

Proof. First we have to prepare several inequalities. Using Lemma 2.10, we choose $0\le\alpha\le\frac12, 2\le q<\infty$ with $2\alpha+\frac{n}{q}=\frac{n}{2}$, and obtain the inequality

$$(4.6)\qquad ||u||_q\le C||A^\alpha u||_2$$

for almost all $t\in[0,T]$ with $C=C(\nu,q,n)>0$. Applying the inequality: $a^{\frac1p}b^{\frac1q}\le\frac{a}{p}+\frac{b}{q}$ with a,b be two nonnegative real numbers and $\frac1p+\frac1q=1,\ p,q\ge1$, then the interpolation inequality

$$(4.7)\qquad ||A^\alpha u||_2\le||Au||_2^\alpha||u||_2^{1-\alpha}, u\in D(A),\ \ 0\le\alpha\le1,$$

yields

$$||A^{\alpha}u||_2 = ||(A^{\frac{1}{2}})^{2\alpha}u||_2 \le ||A^{\frac{1}{2}}u||_2^{2\alpha}||u||_2^{1-2\alpha} \le ||A^{\frac{1}{2}}u||_2 + ||u||_2, 0 \le \alpha \le \frac{1}{2}. \tag{4.8}$$

Using (4.6) with $n = 3, 2 \le q \le 6$, we can obtain

$$||u||_q \le C||A^{\frac{1}{2}}u||_2^{2\alpha}||u||_2^{1-2\alpha} \le C(||A^{\frac{1}{2}}u||_2 + ||u||_2). \tag{4.9}$$

Using the properties of J_k, (see [14],Section 3.4), we can get the relation

$$J_k\overline{u} - \overline{u} = (J_k - J_k^{-1}J_k)\overline{u} = -k^{-1}A^{\frac{1}{2}}J_k\overline{u}, \tag{4.10}$$

and the following estimates

$$\begin{aligned} &||J_k\overline{u}||_2 \le ||\overline{u}||_2, \qquad ||(kI + A^{\frac{1}{2}})^{-1}\overline{u}||_2 \le k^{-1}||\overline{u}||_2, \\ &||A^{\frac{1}{2}}J_k\overline{u}||_2 \le k||\overline{u}||_2. \end{aligned} \tag{4.11}$$

Using (4.8), it is easy to see that

$$||A^{\alpha}J_k\overline{u}||_2 \le ||A^{\frac{1}{2}}J_k\overline{u}||_2^{2\alpha}||J_k\overline{u}||_2^{1-2\alpha} \le k^{2\alpha}||\overline{u}||_2 \tag{4.12}$$

for $k \in \mathbb{N}, 0 \le \alpha \le \frac{1}{2}$.

From (4.12), it can easily be seen that

$$\begin{aligned} ||A^{\alpha}J_k\overline{u}||_{2,s;T'} &\le ||A^{\frac{1}{2}}J_k\overline{u}||_{2,2;T'}^{2\alpha}||J_k\overline{u}||_{2,\infty;T'}^{1-2\alpha} \\ &\le ||A^{\frac{1}{2}}J_k\overline{u}||_{2,2;T'} + ||J_k\overline{u}||_{2,\infty;T'} \end{aligned} \tag{4.13}$$

with $s = \frac{1}{\alpha}, 0 \le \alpha \le \frac{1}{2}, 0 < T' < T$.

Using (4.9) and (4.11), we can get that

$$\begin{aligned} ||J_k\overline{u}||_{q,\infty;T'} &\le C(||A^{\frac{1}{2}}J_k\overline{u}||_{2,\infty;T'} + ||J_k\overline{u}||_{2,\infty;T'}) \\ &\le C(k||\overline{u}||_{2,\infty;T'} + ||\overline{u}||_{2,\infty;T'}) < \infty \end{aligned} \tag{4.14}$$

with q as in (4.9), and it follows that

$$||J_k\overline{u}||_{q,s;T'} < \infty, \tag{4.15}$$

with $s \ge 1, 0 < T' < T$.

This means in particular that the Serrin's condition:

$$J_k\overline{u} \in L^s(0,T;L^q(\Omega)^n), \tag{4.16}$$

are satisfied with certain values q, s such that $n < q < \infty, 2 < s < \infty, \frac{n}{q} + \frac{2}{s} \le 1$.

Next we use $\nu^{\frac{1}{2}}||\nabla u||_2 = ||A^{\frac{1}{2}}u||_2$, and besides, we use (4.6) with $\alpha = \frac{n}{8}$, $q = 4, 2\alpha + \frac{n}{4} = \frac{n}{2}$. This leads to the following inequality:

$$
\begin{aligned}
||(J_k\overline{u}) \otimes w||_2 &\leq C||J_k\overline{u}||_4||w||_4 \\
&\leq C'||A^\alpha J_k\overline{u}||_2||A^\alpha w||_2 \\
&= C'||(A^{\frac{1}{2}})^{2\alpha} J_k\overline{u}||_2||(A^{\frac{1}{2}})^{2\alpha} w||_2 \\
&\leq C'||A^{\frac{1}{2}} J_k\overline{u}||_2^{2\alpha}||J_k\overline{u}||_2^{1-2\alpha}||A^{\frac{1}{2}}w||_2^{2\alpha}||w||_2^{1-2\alpha} \\
&\leq C'k^{2\alpha}||\overline{u}||_2||A^{\frac{1}{2}}w||_2^{2\alpha}||w||_2^{1-2\alpha},
\end{aligned} \tag{4.17}
$$

and therefore, we can get that

$$
\begin{aligned}
||(J_k\overline{u}) \otimes w||_{2,8/n;T'} &\leq C'k^{\frac{n}{4}}||\overline{u}||_{2,\infty;T'}||A^{\frac{1}{2}}w||_{2,2;T'}^{n/4}||w||_{2,\infty;T'}^{1-n/4} \\
&\leq 2C'k^{\frac{n}{4}}\sqrt{E_{T'}(\overline{u})E_{T'}(w)}
\end{aligned} \tag{4.18}
$$

with $0 < T' < T, C = C(\nu, n) > 0, C' = C'(\nu, n) > 0$. It follows that

$$
(J_k\overline{u}) \otimes w \in L^2(0, T; L^2(\Omega)^{3\times 3}), \tag{4.19}
$$

and therefore

$$
\begin{aligned}
\langle (J_k\overline{u}) \cdot \nabla w, w\rangle_{\Omega,T'} = \langle \mathrm{div}((J_k\overline{u}) \otimes w), w\rangle_{\Omega,T'} &= -\frac{1}{2}\left\langle (J_k\overline{u}), \nabla |w|^2\right\rangle_{\Omega,T'} \\
&= \frac{1}{2}\left\langle \mathrm{div}\ (J_k\overline{u}), |w|^2\right\rangle_{\Omega,T'} = 0.
\end{aligned} \tag{4.20}
$$

Using $||A^\alpha J_k u||_2 = ||J_k A^\alpha u||_2 \leq ||A^\alpha u||_2$, (4.8) and (4.6) with $\alpha = \frac{n}{8}$, $q = 4, 2\alpha + \frac{n}{4} = \frac{n}{2}$, we get similarly that

$$
\begin{aligned}
||(J_k\overline{u}) \otimes w||_2 &\leq C||J_k\overline{u}||_4||w||_4 \\
&\leq C'||A^\alpha\overline{u}||_2||A^\alpha w||_2 \\
&\leq C'||A^{\frac{1}{2}}\overline{u}||_2^{\frac{n}{4}}||\overline{u}||_2^{1-\frac{n}{4}}||A^{\frac{1}{2}}w||_2^{\frac{n}{4}}||w||_2^{1-\frac{n}{4}}
\end{aligned} \tag{4.21}
$$

and this leads to

$$
\begin{aligned}
||(J_k\overline{u}) \otimes w||_{2,4/n;T'} &\leq C||A^{\frac{1}{2}}\overline{u}||_{2,2;T'}^{n/4}||\overline{u}||_{2,\infty;T'}^{1-\frac{n}{4}}||A^{\frac{1}{2}}w||_{2,2;T'}^{n/4}||w||_{2,\infty;T'}^{1-\frac{n}{4}} \\
&\leq C\sqrt{E_{T'}(\overline{u})E_{T'}(w)}
\end{aligned} \tag{4.22}
$$

with $0 < T' \leq T$ and $C = C(\nu, n) > 0$ not depending on k.

To prove the property a) we use (4.19), observe that w is also a weak solution of the following linear system

$$
w_t - \nu\Delta w + \nabla p = \overline{g}, \quad \mathrm{div} w = 0, \quad w|_{\partial\Omega} = 0, \quad w(0) = \psi^0
$$

with $\overline{g} = g + L(\overline{w})w + \mathrm{div}((J_k\overline{u}) \otimes w)$, and apply Theorem 2.14. This shows that w is strongly continuous, after a corresponding redefinition, and that

$$
\frac{1}{2}||w(t)||_2^2 + \nu\int_0^t ||\nabla w||_2^2 \mathrm{d}\tau = \frac{1}{2}||\psi^0||_2^2 + \int_0^t \langle g, w\rangle\, \mathrm{d}\tau + \int_0^t \langle L(\overline{w})w, w\rangle\, \mathrm{d}\tau - \int_0^t \langle (J_k\overline{u}) \otimes w, \nabla w\rangle\, \mathrm{d}\tau \tag{4.23}
$$

with $0 \le t \le T$. Using (4.20) we get the energy equality (4.4). This proves a) and c).

From the energy equality (4.4) and the definition of linear operator $L(\overline{w})$ (see (4.2)), we get

(4.24)
$$\sup_{0\le\tau\le t}\left(\frac{1}{2}||w(\tau)||_2^2+\nu\int_0^\tau||\nabla w||_2^2\mathrm{d}\tau\right)\le\frac{1}{2}||\psi^0||_2^2+\int_0^t|\langle g,w\rangle|\mathrm{d}\tau+C_1||\nabla w||_{2,2;t}||w||_{2,2;t}||\overline{w}||_{\infty,\infty;T}$$

with $0 \le t \le T$. And therefore, choosing a positive T_1 small enough, such that $T_1^{\frac{1}{2}}||\overline{w}||_{\infty,\infty;T} \le \min\{\frac{1}{4C_1}, \frac{\nu}{4C_1}\}$, it is clear that

(4.25)
$$\begin{aligned}\frac{1}{2}\sup_{0\le\tau\le T_1}||w(\tau)||_2^2+\nu\int_0^{T_1}||\nabla w||_2^2\mathrm{d}\tau &\le ||\psi^0||_2^2+2\int_0^{T_1}|\langle g,w\rangle|\mathrm{d}\tau\\ &\quad+2||\nabla w||_{2,2;T_1}^2C_1T_1^{\frac{1}{2}}||\overline{w}||_{\infty,\infty;T}+\frac{1}{2}||w||_{2,\infty;T_1}^2C_1T_1^{\frac{1}{2}}||\overline{w}||_{\infty,\infty;T}\\ &\le ||\psi^0||_2^2+2\sup_{0\le\tau\le T_1}||w(\tau)||_2\int_0^{T_1}||g||_2\mathrm{d}\tau\\ &\quad+\frac{\nu}{2}||\nabla w||_{2,2;T_1}^2+\frac{1}{8}||w||_{2,\infty;T_1}^2\\ &\le ||\psi^0||_2^2+\frac{1}{8}\sup_{0\le\tau\le T_1}||w(\tau)||_2^2+8\left(\int_0^{T_1}||g||_2\mathrm{d}\tau\right)^2\\ &\quad+\frac{\nu}{2}||\nabla w||_{2,2;T_1}^2+\frac{1}{8}||w||_{2,\infty;T_1}^2.\end{aligned}$$

It follows that

$$\frac{1}{2}||w||_{2,\infty;T_1}^2+\nu||\nabla w||_{2,2;T_1}^2\le 2||\psi^0||_2^2+16\left(\int_0^{T_1}||g||_2\mathrm{d}\tau\right)^2. \tag{4.26}$$

Then we will repeat this procedure, choosing a positive $T_2 - T_1$ small enough, such that $(T_2 - T_1)^{\frac{1}{2}}||\overline{w}||_{\infty,\infty;T} \le \min\{\frac{1}{4C_1}, \frac{\nu}{4C_1}\}$, we can get

$$\begin{aligned}\frac{1}{2}||w||_{2,\infty;(T_1,T_2)}^2+\nu||\nabla w||_{2,2;(T_1,T_2)}^2 &\le 2||w(T_1)||_2^2+16\left(\int_{T_1}^{T_2}||g||_2\mathrm{d}\tau\right)^2\\ &\le 8||\psi^0||_2^2+64\left(\int_0^{T_1}||g||_2\mathrm{d}\tau\right)^2+16\left(\int_{T_1}^{T_2}||g||_2\mathrm{d}\tau\right)^2\end{aligned} \tag{4.27}$$

Choosing a positive $T_3 - T_2$ small enough, such that $(T_3 - T_2)^{\frac{1}{2}}||\overline{w}||_{\infty,\infty;T} \le \min\{\frac{1}{4C_1}, \frac{\nu}{4C_1}\}$, we can get

$$\begin{aligned}\frac{1}{2}||w||_{2,\infty;(T_2,T_3)}^2+\nu||\nabla w||_{2,2;(T_2,T_3)}^2 &\le 2||w(T_2)||_2^2+16\left(\int_{T_2}^{T_3}||g||_2\mathrm{d}\tau\right)^2\\ &\le 32||\psi^0||_2^2+256\left(\int_0^{T_1}||g||_2\mathrm{d}\tau\right)^2+64\left(\int_{T_1}^{T_2}||g||_2\mathrm{d}\tau\right)^2+16\left(\int_{T_2}^{T_3}||g||_2\mathrm{d}\tau\right)^2.\end{aligned} \tag{4.28}$$

Continuing the above program, after finitely many steps, we can deduce that there is one constant $C_0(\overline{w}, \nu, T)$, such that the energy inequality d) hold.

To prove the integral equation (4.3), noting (4.19) hold, we argue as in the proof of ((1.3.5) of Theorem 1.3.1, [14], P270-271), we only have to replace $div(F - u \otimes u)$ by $div((J_k\overline{u}) \otimes w)$, replace f_0 by $g + L(\overline{w})w$. This proves the lemma. □

The existence result below rests on Banach's fixed point principle.

Lemma 4.2. *Let $\psi^0 \in L^2_\sigma(\Omega)$, $g \in L^1(0,T;L^2(\Omega)^3)$. Then the system (4.1) exists a uniquely determined weak solution*

$$w = w^{(k)} \in L^\infty(0,T;L^2_\sigma(\Omega)) \cap L^2(0,T;W^{1,2}_{0,\sigma}(\Omega)).$$

Proof. The space $X_T := L^\infty(0,T;L^2_\sigma(\Omega)) \cap L^2(0,T;W^{1,2}_{0,\sigma}(\Omega))$ is a Banach space with norm

$$|||w|||_T := ||w||_{2,\infty;T} + ||A^{\frac{1}{2}}w||_{2,2;T} = ||w||_{2,\infty;T} + \nu^{\frac{1}{2}}||\nabla w||_{2,2;T}. \tag{4.29}$$

We will prove the existence of a unique weak solution $w \in X_T$ of (4.1).

Let $w \in X_T$. Set $\hat{F} := (J_k\overline{u}) \otimes w$. Then, we conclude that

$$\hat{w} := F_T(w) := S(\cdot)\psi^0 + \mathcal{J}Pg + \mathcal{J}L(\overline{w})w + A^{\frac{1}{2}}\mathcal{J}A^{-\frac{1}{2}}P\text{div}\hat{F} \tag{4.30}$$

is a weak solution of the linear system

$$\hat{w}_t - \nu\Delta\hat{w} + \nabla p = g + L(\overline{w})w + \text{div}\hat{F},\ \text{div}\hat{w} = 0,\ \hat{w}|_{\partial\Omega} = 0,\ \hat{w}(0) = \psi^0, \tag{4.31}$$

with data $g + L(\overline{w})w + \text{div}\hat{F}$, ψ^0.

Our aim is to find some $w \in X_T$ in such a way that $\hat{w} = F_T(w) = w$.

In the first step we show that there exists some $T', 0 < T' \leq T$, such that $w = F_{T'}(w)$ has a unique solution $w \in X_{T'}$. Then w is a weak solution of (4.31) in $[0,T']$, and therefore also a weak solution of the system (4.1) in $[0,T']$ with data g, ψ^0. Then we will repeat this procedure with $\tilde{w}$ defined by

$$\tilde{w}(t) = w(T'+t),\ t \geq 0,\ \tilde{w}(0) = w(T'). \tag{4.32}$$

This yields the desired solution in the next interval if $T' < T$, and so on. In this way we can get the desired solution on the whole interval $[0,T]$. To solve (4.31), we have to prepare several inequalities. Let $0 < T' \leq T$, and let $w \in X_{T'}$ be given. Noting the following formula

$$\begin{aligned} \big|\, \langle L(\overline{w})w, v\rangle_{\Omega,T'} \big| = \big|\, \langle \nabla w, v \otimes \overline{w}\rangle_{\Omega,T'} \big| &\leq ||\overline{w}||_{\infty,\infty;T'}||\nabla w||_{2,2;T'}||v||_{2,2;T'} \\ &\text{for any}\quad v \in L^2(0,T';L^2_\sigma(\Omega)),\ 0 \leq T' \leq T < +\infty, \end{aligned} \tag{4.33}$$

lead to $L(\overline{w})w \in L^2(0,T';L^2_\sigma(\Omega))$. Furthermore, if let $v = L(\overline{w})w$ in the above formula, then we can deduce that

$$||L(\overline{w})w||_{2,2;T'} \leq ||\overline{w}||_{\infty,\infty;T'}||\nabla w||_{2,2;T'},\ \text{for}\quad 0 \leq T' \leq T < +\infty. \tag{4.34}$$

Since $\hat{w} = F_{T'}(w)$ is a weak solution of (4.31), we argue as in the proof of c) of Theorem 2.14, use (4.34), $||u||^2_{2,1;T'} \le T'||u||^2_{2,2;T'}$, $\nu^{\frac{1}{2}}||\nabla u||_2 = ||A^{\frac{1}{2}}u||_2$ and $(a+b)^2 \le 2a^2 + 2b^2$, obtain the estimates

$$
\begin{aligned}
|||\hat{w}|||_{T'} &= ||\hat{w}||_{2,\infty;T'} + \nu^{\frac{1}{2}}||\nabla \hat{w}||_{2,2;T'} \\
&\le 2(\frac{1}{2}||\hat{w}||^2_{2,\infty;T'} + \nu||\nabla \hat{w}||^2_{2,2;T'})^{\frac{1}{2}} \\
&\le 2(2||\psi^0||_2^2 + 4\nu^{-1}||\hat{F}||^2_{2,2;T'} + 8||g + L(\overline{w})w||^2_{2,1;T'})^{\frac{1}{2}} \\
&\le 4(||\psi^0||_2^2 + \nu^{-1}||\hat{F}||^2_{2,2;T'} + 4||g||^2_{2,1;T'} + 4T'||\overline{w}||^2_{\infty,\infty;T'}||\nabla w||^2_{2,2;T'})^{\frac{1}{2}} \\
&\le 4(||\psi^0||_2 + \nu^{-\frac{1}{2}}||\hat{F}||_{2,2;T'} + 2||g||_{2,1;T'} + 2\nu^{-\frac{1}{2}}T'^{\frac{1}{2}}||\overline{w}||_{\infty,\infty;T'}||A^{\frac{1}{2}}w||_{2,2;T'}) \\
&\le 4(||\psi^0||_2 + \nu^{-\frac{1}{2}}||\hat{F}||_{2,2;T'} + 2||g||_{2,1;T'} + 2\nu^{-\frac{1}{2}}T'^{\frac{1}{2}}||\overline{w}||_{\infty,\infty;T}|||w|||_{T'}).
\end{aligned}
\tag{4.35}
$$

Next we choose $s := \frac{8}{3}, \rho := 8$, then it follows that $\frac{1}{2} = \frac{1}{s} + \frac{1}{\rho}$. We use Hölder's inequality, the inequality $||w||_q \le C||A^\alpha w||_2$ with $q = 4, \alpha = \frac{n}{8}, 2\alpha + \frac{n}{q} = \frac{n}{2}, n = 3$, the interpolation inequality

$$
||A^\alpha w||_2 = ||(A^{\frac{1}{2}})^{2\alpha} w||_2 \le ||A^{\frac{1}{2}}w||_2^{2\alpha}||w||_2^{1-2\alpha},
\tag{4.36}
$$

and the blow inequalities in a slightly modified way:

$$
||J_k w||_2 \le ||w||_2, \qquad ||(A^{\frac{1}{2}})J_k w||_2 \le k||w||_2.
\tag{4.37}
$$

This yields

$$
\begin{aligned}
||(J_k\overline{u}) \otimes w||_{2,2;T'} &\le C_1||J_k\overline{u}||_{4,\rho;T'}||w||_{4,s;T'} \\
&\le C_2||A^\alpha J_k\overline{u}||_{2,\rho;T'}||A^\alpha w||_{2,s;T'} \\
&\le C_2 k^{2\alpha}||\overline{u}||_{2,\rho;T'}||A^{\frac{1}{2}}w||^{2\alpha}_{2,2;T'}||w||^{1-2\alpha}_{2,\infty;T'} \\
&\le C_2 k^{2\alpha}(T')^{\frac{1}{\rho}}||\overline{u}||_{2,\infty;T'}(||A^{\frac{1}{2}}w||_{2,2;T'} + ||w||_{2,\infty;T'}) \\
&\le C_2||\overline{u}||_{2,\infty;T}k^{2\alpha}(T')^{\frac{1}{\rho}}|||w|||_{T'},
\end{aligned}
\tag{4.38}
$$

with $C_1 = C_1(n) > 0, C_2 = C_2(\nu, n) > 0$.

From the estimate above we obtain the inequality

$$
|||\hat{w}|||_{T'} = |||F_{T'}(w)|||_{T'} \le a|||w|||_{T'} + b,
\tag{4.39}
$$

with $a := 4\nu^{-\frac{1}{2}}C_2||\overline{u}||_{2,\infty;T}k^{2\alpha}(T')^{\frac{1}{\rho}} + 8\nu^{-\frac{1}{2}}T'^{\frac{1}{2}}||\overline{w}||_{\infty,\infty;T}$ and $b := 4(||\psi^0||_2 + 2||g||_{2,1;T'})$.

Now let $u, w \in X_{T'}$. Then we see in the same way as above that $\hat{u} - \hat{w} = F_{T'}(u) - F_{T'}(w)$ is a weak solution of the linear system

$$
\begin{aligned}
&(\hat{u} - \hat{w})_t - \nu\Delta(\hat{u} - \hat{w}) + \nabla p = \operatorname{div}((J_k\overline{u}) \otimes (u - w)) + L(\overline{w})(u - w), \\
&\operatorname{div}(\hat{u} - \hat{w}) = 0, \quad (\hat{u} - \hat{w})|_{\partial\Omega} = 0, \quad (\hat{u} - \hat{w})(0) = 0.
\end{aligned}
\tag{4.40}
$$

Using the same estimates as above, we obtain instead of (4.39) that

$$
|||\hat{u} - \hat{w}|||_{T'} = |||F_{T'}(u) - F_{T'}(w)|||_{T'} \le a|||u - w|||_{T'}.
\tag{4.41}
$$

We choose T' with $0 < T' \leq T$ in such a way that

$$a < 1, \tag{4.42}$$

and we consider the equations

$$ay + b = y. \tag{4.43}$$

An elementary calculation shows that

$$y_0 = \frac{b}{1-a} > 0, \tag{4.44}$$

is the root of (4.43).

Consider the closed set

$$D_{T'} := \{u \in X_{T'}; |||u|||_{T'} \leq y_0\}. \tag{4.45}$$

If $u \in D_{T'}$ we conclude with (4.39) that

$$|||F_{T'}(u)|||_{T'} \leq a|||u|||_{T'} + b \leq ay_0 + b = y_0, \tag{4.46}$$

and therefore that $F_{T'}(u) \in D_{T'}$.

From (4.41) we get for $u, w \in D_{T'}$ that $|||F_{T'}(u) - F_{T'}(w)|||_{T'} \leq a|||u - w|||_{T'}$. Since $a < 1$, we may apply Banach's fixed point principle and get a unique solution $w \in D_{T'}$ with $w = F_{T'}(w)$, w being a weak solution of the system (4.1) for $[0, T']$. Noting T' is independently of ψ^0, we can repeat this procedure if $T' < T$, with w replaced by $\tilde{w}$ defined in (4.32). This yields the existence of a weak solution $\tilde{w}(t)$ of (3.11) with $\tilde{w}(0) = w(T')$ in the interval $[0, \min\{2T', T\})$. Now $||\psi^0||_2$ is replaced by $||w(T')||_2$. Note that $w(T')$ is well defined since w is strongly continuous after a corresponding redefinition, see Lemma 4.1, a).

If $2T' < T$, we can repeat this procedure, and so on. After finitely many steps this yields a wek solution w of (4.1) in the whole interval $[0, T]$. To prove the uniqueness of w, we suppose there is another weak solution $u \in X_T$ of (4.1) in $[0, T]$. Using $u = S(\cdot)\psi^0 + \mathcal{J}Pg + A^{\frac{1}{2}}\mathcal{J}A^{-\frac{1}{2}}P\text{div}\hat{F} + \mathcal{J}L(\overline{w})w$, we can conclude that $u = F_T(u)$, and we obtain with T' as above the estimate

$$|||u|||_{T'} = ||u||_{2,\infty;T'} + \nu^{\frac{1}{2}}||\nabla u||_{2,2;T'} \leq 4(||\psi^0||_2 + 2||g||_{2,1;T'}) = b < y_0. \tag{4.47}$$

Then $|||u|||_{T'} \leq y_0$ and $|||w|||_{T'} \leq y_0$. The uniqueness of the fixed point in $D_{T'}$ shows that $u = w$ in $[0, T']$. Repeating this conclusion as above, we see that $u = w$ in $[0, T]$. The proof is completed. □

In the following we use some arguments from Temam's book [19]. The first lemma below will be taken from ([19], Chap.III, 2) without proof. In the second lemma we apply the compactness result to the sequence of approximate weak solutions given by Lemma 4.2. In the proof of this lemma we use an important argument from ([19], Chap.III, (3.38)-(3.39)) concerning the function (4.68) below.

First we introduce some notations on the Fourier transform, see ([19], Chap.III, (2.25)) which is basic for our functional analytic approach to the Navier-Stokes system. In order to use the Fourier transform, we need to leave the real vector spaces, which we considered up to now, and to work in the corresponding complexifications of these spaces. We will do it below keeping the same notations as in the real case.

Let $0 < T < \infty$, and let X be a (complex) Hilbert space with scalar product $\langle \cdot, \cdot \rangle_X$ and norm $|| \cdot ||_X$. Then we consider the (complex) Hilbert space $L^2(0, T; X)$ with scalar product $\langle u, v \rangle_{X,T} := \int_0^T \langle u, v \rangle_X \, \mathrm{d}t$, norm $||u||_{X,T} := (\int_0^T ||u||_X^2 \mathrm{d}t)^{\frac{1}{2}}$ and correspondingly the Hilbert space $L^2(\mathbb{R}, X)$ with scalar product $\langle u, v \rangle_{X,\mathbb{R}} := \int_{-\infty}^{\infty} \langle u, v \rangle_X \, \mathrm{d}t$ and norm $||u||_{X,\mathbb{R}} := (\int_{-\infty}^{\infty} ||u||_X^2 \mathrm{d}t)^{\frac{1}{2}}$.

Let $v \in L^2(0, T; X)$. Then it is convenient to extend v by zero to get a function from $\mathbb{R}$ to X. Thus we define the Fourier transform $\mathscr{F}[v]$ of v by

$$\mathscr{F}[v](\tau) = \int_{-\infty}^{\infty} v(t) e^{-2\pi i \tau t} dt = \int_0^T v(t) e^{-2\pi i \tau t} \mathrm{d}t, \qquad \tau \in \mathbb{R}. \tag{4.48}$$

This definition can be extended to a class of distributions in $\mathbb{R}$ in the same way as for scalar functions. In particular we get the important Parseval equality

$$\int_{-\infty}^{\infty} ||\mathscr{F}[v](\tau)||_X^2 \mathrm{d}\tau = \int_0^T ||v(t)||_X^2 \mathrm{d}t \tag{4.49}$$

The following compactness lemma is a special case of ([19], Chap. III, Theorem 2.2), which will play an important role.

Lemma 4.3. *Let X_0, X be Hilbert spaces with norms $|| \cdot ||_{X_0}$ and $|| \cdot ||_X$, respectively, and suppose that there is a compact embedding*

$$X_0 \subseteq X.$$

Let $0 < T < \infty, 0 < \gamma \leq 1$, and let $(v_j)_{j=1}^{\infty}$ be a sequence in $L^2(0, T; X_0)$ satisfying

$$\sup_j \Big(\int_0^T ||v_j||_{X_0}^2 \mathrm{d}t \Big) < \infty, \quad \sup_j \Big(\int_{-\infty}^{\infty} |\tau|^{2\gamma} ||\mathscr{F}[v_j](\tau)||_X^2 \mathrm{d}\tau \Big) < \infty. \tag{4.50}$$

Then there exists a subsequence of $(v_j)_{j=1}^{\infty}$ which converges strongly in $L^2(0, T; X)$ to some $v \in L^2(0, T; X)$.

Proof. See Temam ([19], Chap. III, Theorem 2.2). □

This lemma is needed in the following special situation.

Lemma 4.4. *Let $0 < T < \infty, \psi^0 \in L^2_\sigma(\Omega)$, and let $g \in L^1(0, T; L^2(\Omega)^3)$. Let $w = w^{(k)}, k \in \mathbb{N}$, be the (uniquely determined) weak solution of the approximate Navier-Stokes system (4.1) with data g, ψ^0, (see Lemma 4.2). Then the sequence $w = w^{(k)}, k \in \mathbb{N}$ has the following properties:*

a) If Ω is bounded, then there exists a subsequence which converges strongly in $L^2(0, T; L^2_\sigma(\Omega))$.

b) If $\Omega = \mathbb{R}^3$, then for each bounded Lipschitz subdomain $\Omega_0 \subseteq \Omega$ with $\overline{\Omega_0} \subseteq \Omega$, there exists a subsequence which converges strongly in $L^2(0, T; L^2(\Omega_0)^3)$.

Proof. We know, see Lemma 4.1, that each $w^{(k)} : [0,T] \to L^2_\sigma(\Omega)$ is strongly continuous, and that the integral equation

$$w^{(k)} = S(\cdot)\psi^0 + \mathcal{J}Pg + \mathcal{J}L(\overline{w})w^{(k)} + A^{\frac{1}{2}}\mathcal{J}A^{-\frac{1}{2}}P\mathrm{div}(J_k\overline{u} \otimes w^{(k)}) \tag{4.51}$$

is satisfied. We set $V_1 := S(\cdot)\psi^0, V_2 := \mathcal{J}Pg, V := V_1 + V_2$ and

$$U_k := \mathcal{J}L(\overline{w})w^{(k)} + A^{\frac{1}{2}}\mathcal{J}A^{-\frac{1}{2}}P\mathrm{div}(J_k\overline{u} \otimes w^{(k)}). \tag{4.52}$$

Thus we get the representation

$$w^{(k)} = V + U_k, \qquad k \in \mathbb{N}. \tag{4.53}$$

In the following, $C, C', C_1, ...$ are always positive constants which may depending on $\widetilde{u}, \psi^0, f, T, ...$ but not on $k \in \mathbb{N}$.

Using (2.53) of Theorem 2.15, (2.57) of Theorem 2.16, and again (2.62) of Theorem 2.17, we see that

$$E_T(V) = \frac{1}{2}||V||^2_{2,\infty;T} + \nu||\nabla V||^2_{2,2;T} \le C < \infty. \tag{4.54}$$

Noting the weak solution $w^{(k)}$ satisfies the energy inequality (4.5), letting $T' \to T$, we can obtain that

$$\begin{aligned} E_T(w^{(k)}) &= \frac{1}{2}||w^{(k)}||^2_{2,\infty;T} + \nu||\nabla w^{(k)}||^2_{2,2;T} \\ &\le C_0(\overline{w},\nu,T)||\psi^0||^2_2 + C_0(\overline{w},\nu,T)||g||^2_{2,1;T} < \infty. \end{aligned} \tag{4.55}$$

In addition, inequality (4.18), formula $||L(\overline{w})w^{(k)}||_{2,2;T} \le ||\overline{w}||_{\infty,\infty;T}||\nabla w^{(k)}||_{2,2;T}$ and (4.5) yields

$$||L(\overline{w})w^{(k)}||_{2,s;T} < \infty, \quad ||J_k\overline{u} \otimes w^{(k)}||_{2,s;T} \le Ck^{\frac{n}{4}}\sqrt{E_T(\overline{u})E_T(w^{(k)})} < \infty \tag{4.56}$$

with $1 \le s \le 2, k \in \mathbb{N}, C = C(\nu,n,T) > 0$. In particular we may set $s = 2$.

Thus, by the superposition principle of linear equations, we can apply Theorem 2.17 and Theorem 2.16, (with function $Pf + PdivF$ replaced by $L(\overline{w})w^{(k)} + Pdiv((J_k\overline{u}) \otimes w^{(k)})$) and see that $A^{-\frac{1}{2}}U_k : [0,T] \to L^2_\sigma(\Omega)$ is strongly continuous, that

$$(A^{-\frac{1}{2}}U_k)_t \in L^2(0,T;L^2_\sigma(\Omega)), \quad A^{\frac{1}{2}}U_k \in L^2(0,T;L^2_\sigma(\Omega)), \quad (A^{-\frac{1}{2}}U_k)(0) = 0,$$

and that the evolution equation

$$(A^{-\frac{1}{2}}U_k)_t + A^{\frac{1}{2}}U_k = A^{-\frac{1}{2}}L(\overline{w})w^{(k)} + A^{-\frac{1}{2}}P\mathrm{div}(J_k\overline{u} \otimes w^{(k)}) \tag{4.57}$$

is satisfied. Noting $A^{-\frac{1}{2}}P\mathrm{div} : L^2(\Omega)^{n\times n} \to L^2_\sigma(\Omega)$ be a bounded operator with operator norm (see [14],P154): $||A^{-\frac{1}{2}}P\mathrm{div}|| \le \nu^{-\frac{1}{2}}$, we can get that

$$\begin{aligned} ||A^{-\frac{1}{2}}P\mathrm{div}(J_k\overline{u} \otimes w^{(k)})||_{2,2;T} &\le \nu^{-\frac{1}{2}}||(J_k\overline{u}) \otimes w^{(k)}||_{2,2;T} \\ &\le \nu^{-\frac{1}{2}}Ck^{\frac{n}{4}}\sqrt{E_T(\overline{u})E_T(w^{(k)})} < \infty. \end{aligned} \tag{4.58}$$

Applying the Fourier transform to (4.57), using integration by parts and that $A^{-\frac{1}{2}}U_k(0)=0$, we get that

$$\begin{aligned}\mathscr{F}[U_k](\tau)&=\int_0^T U_k(t)e^{-2\pi i\tau t}\mathrm{d}t,\\ \mathscr{F}[(A^{-\frac{1}{2}}U_k)_t](\tau)&=\int_0^T (A^{-\frac{1}{2}}U_k)_t(t)e^{-2\pi i\tau t}\mathrm{d}t\\ &=A^{-\frac{1}{2}}U_k(T)e^{-2\pi i\tau T}+2\pi i\tau A^{-\frac{1}{2}}\mathscr{F}[U_k](\tau),\end{aligned}\tag{4.59}$$

$\tau\in\mathbb{R}$, and therefore we obtain

$$\begin{aligned}&2\pi i\tau A^{-\frac{1}{2}}\mathscr{F}[U_k](\tau)+A^{\frac{1}{2}}\mathscr{F}[U_k](\tau)\\ &\quad=\mathscr{F}[A^{-\frac{1}{2}}P\mathrm{div}((J_k\overline{u})\otimes w^{(k)})](\tau)+\mathscr{F}[A^{-\frac{1}{2}}L(\overline{w})w^{(k)}](\tau)-A^{-\frac{1}{2}}U_k(T)e^{-2\pi i\tau T}.\end{aligned}\tag{4.60}$$

Taking the scalar product with $A^{\frac{1}{2}}\mathscr{F}[U_k](\tau)$, noting $||L(\overline{w})w^{(k)}||_{2,2;T}\le||\overline{w}||_{\infty,\infty;T}||\nabla w^{(k)}||_{2,2;T}$, (2.8), (4.5) and $A^{-\frac{1}{2}}P\mathrm{div}:L^2(\Omega)^{n\times n}\to L^2_\sigma(\Omega)$ be a bounded operator with operator norm: $||A^{-\frac{1}{2}}P\mathrm{div}||\le\nu^{-\frac{1}{2}}$, it following that

$$\begin{aligned}2\pi|\tau|||\mathscr{F}[U_k](\tau)||_2^2&\le||A^{\frac{1}{2}}\mathscr{F}[U_k](\tau)||_2^2+|\left\langle A^{-\frac{1}{2}}U_k(T),A^{\frac{1}{2}}\mathscr{F}[U_k](\tau)\right\rangle_\Omega|\\ &\quad+|\left\langle\mathscr{F}[A^{-\frac{1}{2}}P\mathrm{div}((J_k\overline{u})\otimes w^{(k)})](\tau),A^{\frac{1}{2}}\mathscr{F}[U_k](\tau)\right\rangle_\Omega|\\ &\quad+|\left\langle\mathscr{F}[A^{-\frac{1}{2}}L(\overline{w})w^{(k)}](\tau),A^{\frac{1}{2}}\mathscr{F}[U_k](\tau)\right\rangle_\Omega|\\ &\le||A^{\frac{1}{2}}\mathscr{F}[U_k](\tau)||_2^2+||A^{-\frac{1}{2}}U_k(T)||_2||A^{\frac{1}{2}}\mathscr{F}[U_k](\tau)||_2\\ &\quad+(\int_0^T||A^{-\frac{1}{2}}P\mathrm{div}(J_k\overline{u}\otimes w^{(k)})||_2\mathrm{d}t)||A^{\frac{1}{2}}\mathscr{F}[U_k](\tau)||_2\\ &\quad+(\int_0^T||A^{-\frac{1}{2}}L(\overline{w})w^{(k)}||_2\mathrm{d}t)||A^{\frac{1}{2}}\mathscr{F}[U_k](\tau)||_2\\ &\le||A^{\frac{1}{2}}\mathscr{F}[U_k](\tau)||_2^2+||A^{-\frac{1}{2}}U_k(T)||_2||A^{\frac{1}{2}}\mathscr{F}[U_k](\tau)||_2\\ &\quad+\nu^{-\frac{1}{2}}(\int_0^T||(J_k\overline{u})\otimes w^{(k)}||_2\mathrm{d}t)||A^{\frac{1}{2}}\mathscr{F}[U_k](\tau)||_2\\ &\quad+C(\overline{w},\nu,T)||\overline{w}||_{\infty,\infty;T}(||\psi^0||_2+||g||_{2,1;T})||A^{\frac{1}{2}}\mathscr{F}[U_k](\tau)||_2.\end{aligned}\tag{4.61}$$

Using the definition (4.52) of U_k, we can get that

$$A^{-\frac{1}{2}}U_k(T)=\int_0^T S(T-\tau)A^{-\frac{1}{2}}P\mathrm{div}((J_k\overline{u})\otimes w^{(k)})\mathrm{d}\tau,\tag{4.62}$$

and therefore, with (4.58) we obtain that

$$\begin{aligned}||A^{-\frac{1}{2}}U_k(T)||_2&\le\int_0^T||A^{-\frac{1}{2}}P\mathrm{div}((J_k\overline{u})\otimes w^{(k)})||_2\mathrm{d}\tau\\ &\le\nu^{-\frac{1}{2}}\int_0^T||((J_k\overline{u})\otimes w^{(k)})||_2\mathrm{d}\tau\end{aligned}\tag{4.63}$$

From (4.22), we can get

$$\|(J_k\overline{u})\otimes w^{(k)}\|_{2,1;T} \le CE_T(w^{(k)}) \tag{4.64}$$

with $C = C(\nu, n, T) > 0$. Using (4.55) we see that

$$\|(J_k\overline{u})\otimes w^{(k)}\|_{2,1;T} \le C \tag{4.65}$$

with $C > 0$ not depending on k. This leads to

$$\|A^{-\frac{1}{2}}U_k(T)\|_2 \le C. \tag{4.66}$$

Thus, from above we now get

$$2\pi|\tau|\|\mathscr{F}[U_k](\tau)\|_2^2 \le \|A^{\frac{1}{2}}\mathscr{F}[U_k](\tau)\|_2^2 + C\|A^{\frac{1}{2}}\mathscr{F}[U_k](\tau)\|_2 \tag{4.67}$$

with $C > 0$ not depending on k.

Let $0 < \gamma < \frac{1}{4}$, then an elementary calculation shows that

$$|\tau|^{2\gamma} \le C'(\frac{1+|\tau|}{1+|\tau|^{1-2\gamma}}) \tag{4.68}$$

holds for all $\tau \in \mathbb{R}$ with $C' > 0$ not depending on τ. We use this estimate in a similar way as in ([19], Chap. III, (3.38)-(3.39)). By (4.67) and (4.49), we can obtain that

$$\begin{aligned}
\int_{-\infty}^{+\infty} |\tau|^{2\gamma}\|\mathscr{F}[U_k](\tau)\|_2^2 \mathrm{d}\tau &\le C'\int_{-\infty}^{+\infty}(\frac{1+|\tau|}{1+|\tau|^{1-2\gamma}})\|\mathscr{F}[U_k](\tau)\|_2^2\mathrm{d}\tau \\
&\le C'\int_{-\infty}^{+\infty}\|\mathscr{F}[U_k](\tau)\|_2^2\mathrm{d}\tau \\
&\qquad + C'\int_{-\infty}^{+\infty}(\frac{|\tau|}{1+|\tau|^{1-2\gamma}})\|\mathscr{F}[U_k](\tau)\|_2^2\mathrm{d}\tau \\
&\le C'\int_0^T \|U_k(t)\|_2^2\mathrm{d}t + C'\int_0^T \|A^{\frac{1}{2}}U_k(t)\|_2^2\mathrm{d}t \\
&\qquad + C'\int_{-\infty}^{+\infty}(\frac{1}{1+|\tau|^{1-2\gamma}})\|A^{\frac{1}{2}}\mathscr{F}[U_k](\tau)\|_2\mathrm{d}\tau
\end{aligned} \tag{4.69}$$

We can get from (4.54), (4.55) that

$$\|U_k\|_{2,2;T}^2 \le C, \quad \nu\|\nabla U_k\|_{2,2;T}^2 = \|A^{\frac{1}{2}}U_k\|_{2,2;T}^2 \le C, \tag{4.70}$$

since $U_k = w^{(k)} - V$. In addition, it is clear that

$$\int_{-\infty}^{+\infty}(\frac{1}{1+|\tau|^{1-2\gamma}})^2\mathrm{d}\tau < \infty, \tag{4.71}$$

since $0<\gamma<\frac{1}{4},\ \ 2(1-2\gamma)>1$. This yields

$$\begin{aligned}\int_{-\infty}^{+\infty}|\tau|^{2\gamma}||\mathscr{F}[U_k](\tau)||_2^2\mathrm{d}\tau &\le C_1+C_2\\ &+C_3\Big(\int_{-\infty}^{+\infty}\big(\frac{1}{1+|\tau|^{1-2\gamma}}\big)^2\mathrm{d}\tau\Big)^{\frac{1}{2}}\Big(\int_{-\infty}^{+\infty}||A^{\frac{1}{2}}\mathscr{F}[U_k]||_2^2\mathrm{d}\tau\Big)^{\frac{1}{2}}\\ &=C_4+C_5\Big(\int_0^T||A^{\frac{1}{2}}U_k(t)||_2^2\mathrm{d}t\Big)^{\frac{1}{2}}\le C_6,\end{aligned}\tag{4.72}$$

with constants $C, C', C_1, ..., C_6$ not depending on $k\in\mathbb{N}$.

Therefore, we obtain that

$$\sup_k\Big(\int_0^T||\nabla U_k||_2^2\mathrm{d}t\Big)<\infty,\quad \sup_k\Big(\int_{-\infty}^{+\infty}|\tau|^{2\gamma}||\mathscr{F}[U_k](\tau)||_2^2\mathrm{d}\tau\Big)<\infty.\tag{4.73}$$

Consider any bounded Lipschitz subdomain $\Omega_0\subseteq\Omega$ with $\overline{\Omega_0}\subseteq\Omega$. Then, we have

$$\begin{aligned}\mathscr{F}[U_k](\tau,x)&=\int_0^T U_k(t,x)e^{-2\pi i\tau t}\mathrm{d}t,\tau\in\mathbb{R},x\in\Omega,\\ ||\mathscr{F}[U_k](\tau)||_{L^2(\Omega_0)}&=\int_{\Omega_0}|\mathscr{F}[U_k](\tau,x)|^2dx\le\int_\Omega|\mathscr{F}[U_k](\tau,x)|^2dx,\end{aligned}\tag{4.74}$$

and it leads to

$$\sup_k\Big(\int_{-\infty}^{+\infty}|\tau|^{2\gamma}||\mathscr{F}[U_k](\tau)||_{L^2(\Omega_0)}^2\mathrm{d}\tau\Big)\le\sup_k\Big(\int_0^T|\tau|^{2\gamma}||\mathscr{F}[U_k](\tau)||_2^2\mathrm{d}\tau\Big)<\infty.\tag{4.75}$$

Further we obtain

$$\begin{aligned}\sup_k\Big(\int_0^T||\nabla U_k(t)||_{L^2(\Omega_0)}^2\mathrm{d}t\Big)&=\sup_k\Big(\int_0^T\int_{\Omega_0}|\nabla U_k(t,x)|^2dxdt\Big)\\ &\le\sup_k\Big(\int_0^T\int_\Omega|\nabla U_k(t,x)|^2dxdt\Big)<\infty.\end{aligned}\tag{4.76}$$

If Ω is a bounded domain, by Rellich's Theorem, we know that the embedding

$$X_0\subseteq X,\ X_0:=W_0^{1,2}(\Omega)^n,\ X:=L^2(\Omega)^n,\ \text{is compact.}$$

From (4.73), we see that sequence $(U_k)_{k=1}^\infty$ satisfies the above condition (4.50). Then Lemma 4.3 yields the existence of a subsequence which converges strongly in $L^2(0,T;L^2_\sigma(\Omega))$. The sequence $(w^{(k)})_{k=1}^\infty$ has this property too, since $w^{(k)}=U_k+V,\ V\in L^2(0,T;W^{1,2}_{0,\sigma}(\Omega))$.

If $\Omega=\mathbb{R}^3$, we can choose $X_0:=W_0^{1,2}(\Omega_0)^n, X:=L^2(\Omega_0)^n$, and get by Rellich's Theorem, the compact embedding $W_0^{1,2}(\Omega_0)^n\subseteq L^2(\Omega_0)^n$. Consider $(U_k)_{k=1}^\infty$ as a sequence in $L^2(\Omega_0)^n$. Then, from (4.75), (4.76) we get the validity of (4.50), and Lemma 4.3 shows the existence of a subsequence satisfying the property in Lemma 4.4, b). The proof is complete. □

We are now in a position to prove Lemma 3.6.

Proof. To construct a weak solution w as required in this lemma, let $(w^{(k)})_{k=1}^{\infty}$ be the sequence of solutions of the approximate system as in Lemma 4.2. We investigate the convergence properties as $k \to \infty$.

In order to investigate the convergence property of the weak solution $(w^{(k)})_{k=1}^{\infty}$ as $k \to +\infty$. First let $T' \to T$ in the energy estimate (4.5) and obtain the inequality

$$\frac{1}{2}||w^{(k)}||_{2,\infty;T}^2 + \nu||\nabla w^{(k)}||_{2,2;T}^2 \leq C_0(\overline{w},\nu,T)||\psi^0||_2^2 + C_0(\overline{w},\nu,T)||g||_{2,1;T}^2. \tag{4.77}$$

In particular, from the energy inequality (4.77), it follows that $(w^{(k)})_{k=1}^{\infty}$ is a bounded sequence in the Hilbert space $L^2(0,T;W_{0,\sigma}^{1,2}(\Omega))$. Since this space is reflexive, we find a subsequence which converges weakly in this space to some element $w \in L^2(0,T;W_{0,\sigma}^{1,2}(\Omega))$. For simplicity we may assume that the sequence $(w^{(k)})_{k=1}^{\infty}$ itself has this property.

Let Ω be bounded. Then, from Lemma 4.4, we know that the existence of a subsequence of $(w^{(k)})_{k=1}^{\infty}$ which converges to w strongly in $L^2(0,T;L_\sigma^2(\Omega))$. Again we may assume that the sequence $(w^{(k)})_{k=1}^{\infty}$ itself has this property. In addition, The Fischer-Riez theorem (see [[2], Note at the end of Chapter 10.25]), yields the existence of a subsequence which converges strongly to $w(t)$ for almost all $t \in [0,T)$. We may assume that the sequence itself has this property. Therefore, we get the existence of a null set $\mathcal{N} \subseteq [0,T)$ such that

$$w(t) = s - \lim_{k\to\infty} w^{(k)}(t) \tag{4.78}$$

for all $t \in [0,T)\backslash\mathcal{N}$. We thus obtain the following convergence properties of the sequence $(w^{(k)})_{k=1}^{\infty}$:

$$\begin{cases} (w^{(k)})_{k=1}^{\infty}\text{converges to } w \text{ weakly in } L^2(0,T;W_{0,\sigma}^{1,2}(\Omega)) \text{ and strongly in } L^2(0,T;L_\sigma^2(\Omega)); \\ (\nabla w^{(k)})_{k=1}^{\infty} \text{ converges to } \nabla w \text{ weakly in } L^2(0,T;L^2(\Omega)^{3\times 3}); \\ (w^{(k)}(t))_{k=1}^{\infty} \text{ converges to } w(t) \text{ strongly in } L_\sigma^2(\Omega) \text{ for all } t \in [0,T)\backslash\mathcal{N}. \end{cases} \tag{4.79}$$

Let's consider any test function $v \in C_0^\infty([0,T);C_{0,\sigma}^\infty(\Omega))$. Since $w^{(k)}$ is the weak solution of the approximate Navier-Stokes system (4.1), from the definition of the weak solution, we can get that

$$\begin{aligned} -\left\langle w^{(k)}, v_t\right\rangle_{\Omega,T} + \nu\left\langle \nabla w^{(k)}, \nabla v\right\rangle_{\Omega,T} - \left\langle (J_k\overline{u})\cdot\nabla w^{(k)}, v\right\rangle_{\Omega,T} - \left\langle \nabla w^{(k)}, v\otimes\overline{w}\right\rangle_{\Omega,T} \\ = \left\langle \psi^0, v(0)\right\rangle_\Omega + \langle g, v\rangle_{\Omega,T}. \end{aligned} \tag{4.80}$$

Using (4.79), it can easily be seen that

$$\langle w, v_t\rangle_{\Omega,T} = \lim_{k\to\infty}\left\langle w^{(k)}, v_t\right\rangle_{\Omega,T} \tag{4.81}$$

and

$$\langle \nabla w, \nabla v\rangle_{\Omega,T} = \lim_{k\to\infty}\left\langle \nabla w^{(k)}, \nabla v\right\rangle_{\Omega,T}, \quad \langle \nabla w, v\otimes\overline{w}\rangle_{\Omega,T} = \lim_{k\to\infty}\left\langle \nabla w^{(k)}, v\otimes\overline{w}\right\rangle_{\Omega,T}. \tag{4.82}$$

In order to treat the limit of $\left\langle (J_k\widetilde{u})\cdot\nabla w^{(k)}, v\right\rangle_{\Omega,T}$, we write

$$\left\langle (J_k\overline{u})\cdot\nabla w^{(k)}, v\right\rangle_{\Omega,T} = -\left\langle (J_k\overline{u})\otimes w^{(k)}, \nabla v\right\rangle_{\Omega,T}, \tag{4.83}$$

and obtain the representation

$$\begin{aligned}&\left\langle (J_k\overline{u})\otimes w^{(k)}, \nabla v\right\rangle_{\Omega,T} - \langle \overline{u}\otimes w, \nabla v\rangle_{\Omega,T}\\ &\quad= \left\langle (J_k\overline{u})\otimes (w^{(k)}-w), \nabla v\right\rangle_{\Omega,T} + \langle ((J_k-I)\overline{u})\otimes w, \nabla v\rangle_{\Omega,T}.\end{aligned} \tag{4.84}$$

Using the Hӧldr's inequality and inequality: $||(J_k\overline{u})||_{2,2;T}\le ||\overline{u}||_{2,2;T}$, we can get

$$\begin{aligned}|\left\langle (J_k\overline{u})\otimes (w^{(k)}-w), \nabla v\right\rangle_{\Omega,T}| &\le ||(J_k\overline{u})\otimes (w^{(k)}-w)||_{1,1;T}||\nabla v||_{\infty,\infty;T}\\ &\le C||J_k\overline{u}||_{2,2;T}||w^{(k)}-w||_{2,2;T}||\nabla v||_{\infty,\infty;T}\\ &\le C||\overline{u}||_{2,2;T}||w^{(k)}-w||_{2,2;T}||\nabla v||_{\infty,\infty;T},\end{aligned} \tag{4.85}$$

and

$$|\langle ((J_k-I)\overline{u})\otimes w, \nabla v\rangle_{\Omega,T}| \le C||((J_k-I)\overline{u})||_{2,2;T}||w||_{2,2;T}||\nabla v||_{\infty,\infty;T} \tag{4.86}$$

with $C=C(n)>0$ not depending on T. Combining (4.79) and (4.85), we can obtain

$$\lim_{k\to\infty}\left\langle (J_k\overline{u})\otimes (w^{(k)}-w), \nabla v\right\rangle_{\Omega,T} = 0. \tag{4.87}$$

Using the properties of Yosida approximation operators $J_k, k\in\mathbb{N}$ (see (3.4.8) of [14], P105), we get that

$$\lim_{k\to\infty}||((J_k-I)\overline{u})||_2 = 0, \text{ for all } t\in[0,T)\backslash\mathcal{N}. \tag{4.88}$$

Further we obtain

$$||(J_k-I)\overline{u}(t)||_2 \le ||J_k\overline{u}(t)||_2 + ||\overline{u}(t)||_2 \le 2||\overline{u}(t)||_2 \tag{4.89}$$

for almost all $t\in[0,T)$. Therefore, we may use Lebesgue's dominated convergence lemma, and get

$$\lim_{k\to\infty}||(J_k-I)\overline{u}||_{2,2;T} = 0. \tag{4.90}$$

Combining (4.86) and (4.90), it follows that

$$\lim_{k\to\infty}\langle ((J_k-I)\overline{u})\otimes w, \nabla v\rangle_{\Omega,T} = 0. \tag{4.91}$$

Thus, by (4.83),(4.84),(4.87) and (4.91), we may let $k\to\infty$ in each term of (4.80), and obtain that

$$-\langle w, v_t\rangle_{\Omega,T} + \nu\langle \nabla w, \nabla v\rangle_{\Omega,T} + \langle \overline{u}\otimes w, \nabla v\rangle_{\Omega,T} - \langle L(\overline{w})w, v\rangle_{\Omega,T} = \left\langle \psi^0, v(0)\right\rangle_{\Omega} + \langle g, v\rangle_{\Omega,T}. \tag{4.92}$$

This shows that w is a weak solution of the Navier-Stokes system (3.11).

To prove c) of Lemma 3.6, let's use the Lemma 4.1 and conclude that each $w^{(k)}$ is strongly continuous, after a corresponding redefinition on $[0,T)$, and that

$$\frac{1}{2}||w^{(k)}(t)||_2^2+\nu\int_0^t||\nabla w^{(k)}||_2^2\mathrm{d}\tau=\frac{1}{2}||\psi^0||_2^2+\int_0^t\left\langle g,w^{(k)}\right\rangle_\Omega\mathrm{d}\tau+\int_0^t\left\langle\nabla w^{(k)},w^{(k)}\otimes\overline{w}\right\rangle\mathrm{d}\tau \tag{4.93}$$

for all $t\in[0,T)$. The weak convergence property in (4.79) concerning $(\nabla w^{(k)})_{k=1}^\infty$ shows that

$$||\nabla w||_{2,2;t}\le\lim_{k\to\infty}\inf||\nabla w^{(k)}||_{2,2;t} \tag{4.94}$$

for all $t\in[0,T)$, and the property concerning $(w^{(k)}(t))_{k=1}^\infty$ shows that

$$||w(t)||_2^2=\lim_{k\to\infty}||w^{(k)}(t)||_2^2 \tag{4.95}$$

for all $t\in[0,T)\backslash\mathcal{N}$. The properties in (4.79) also show that the following equation

$$\int_0^t\langle g,w\rangle\,\mathrm{d}\tau=\lim_{k\to\infty}\int_0^t\left\langle g,w^{(k)}\right\rangle\mathrm{d}\tau, \tag{4.96}$$

hold, for all $t\in[0,T)$.

Besides, using (4.5), it follows that

$$\begin{aligned}|\int_0^t\left\langle\nabla w^{(k)},(w^{(k)}-w)\otimes\overline{w}\right\rangle\mathrm{d}\tau|&\le||\nabla w^{(k)}||_{2,2;t}||w^{(k)}-w||_{2,2;t}||\overline{w}||_{\infty,\infty;T}\\&\le\nu^{-\frac{1}{2}}C_0^{\frac{1}{2}}(\overline{w},\nu,T)(||\psi^0||_2+||g||_{2,1;T})||w^{(k)}-w||_{2,2;t}||\overline{w}||_{\infty,\infty;T}\end{aligned} \tag{4.97}$$

noting the following representation formula

$$\int_0^t\left\langle\nabla w^{(k)},w^{(k)}\otimes\overline{w}\right\rangle\mathrm{d}\tau=\int_0^t\left\langle\nabla w^{(k)},w\otimes\overline{w}\right\rangle\mathrm{d}\tau+\int_0^t\left\langle\nabla w^{(k)},(w^{(k)}-w)\otimes\overline{w}\right\rangle\mathrm{d}\tau, \tag{4.98}$$

combining the two formulas above and (4.79), we can obtain

$$\lim_{k\to\infty}\int_0^t\left\langle\nabla w^{(k)},w^{(k)}\otimes\overline{w}\right\rangle\mathrm{d}\tau=\int_0^t\langle\nabla w,w\otimes\overline{w}\rangle\,\mathrm{d}\tau \tag{4.99}$$

Taking $\lim\inf_{k\to\infty}$ in each term of (4.93), we get the following energy inequality

$$\frac{1}{2}||w(t)||_2^2+\nu\int_0^t||\nabla w||_2^2\mathrm{d}\tau\le\frac{1}{2}||\psi^0||_2^2+\int_0^t\langle g,w\rangle_\Omega\,\mathrm{d}\tau+\int_0^t\langle\nabla w,w\otimes\overline{w}\rangle\,\mathrm{d}\tau \tag{4.100}$$

for all $t\in[0,T)\backslash\mathcal{N}$.

From the Lemma 3.5, we can conclude that $w:[0,T]\to L^2_\sigma(\Omega)$ is weakly continuous, after a corresponding redefinition on $[0,T)$. Therefore, for each $t\in[0,T]$ we find a sequence $(t_j)_{j=1}^\infty$ in $[0,T)\backslash\mathcal{N}$ so that $w(t_j)$ tends to $w(t)$, weakly in $L^2_\sigma(\Omega)$ as $j\to\infty$. It follows that

$$||w(t)||_2\le\lim_{j\to\infty}\inf||w(t_j)||_2. \tag{4.101}$$

Inserting $t = t_j$ in (4.100) and taking $\liminf_{j\to\infty}$ in each term, we can conclude that (4.100) holds for all $t \in [0, T]$.

The inequality d) is a consequence of (4.100), which has been shown in the proof of Lemma 4.1, see (4.5). To prove b), we use the same argument as used for b) of Lemma 4.1.

This proves Lemma 3.6 for the case that Ω is bounded, that $0 < T < \infty$, and that $\psi^0 \in L^2_\sigma(\Omega), g \in L^1(0, T; L^2(\Omega)^n)$.

Consider now the case: $\Omega = \mathbb{R}^3$.

By lemma 2.12, we can choose a sequence $(\Omega_j)_{j=1}^\infty$ of bounded Lipschitz subdomains of $\Omega_j \subseteq \Omega$ with $\overline{\Omega_j} \subseteq \Omega_{j+1}, j \in \mathbb{N}$, and $\Omega = \bigcup_{j=1}^\infty \Omega_j$.

For each bounded Lipschitz subdomain Ω_0 with $\overline{\Omega_0} \subset \Omega$, we can find some $j \in \mathbb{N}$ so that $\Omega_0 \subseteq \Omega_j$.

In the next step, we will construct, for each $j \in \mathbb{N}$, a subsequence $(w^{(k)}_{(j)})_{k=1}^\infty$ of $(w^{(k)})_{k=1}^\infty$, a function $w_{(j)} \in L^2(0, T; W^{1,2}(\Omega_j)^3)$, and a null set $\mathcal{N}_j \subseteq [0, T]$ with the following properties corresponding to

(4.102)
$$\begin{cases} (w^{(k)}_{(j)})_{k=1}^\infty \text{converges to } w_{(j)} \text{ weakly in } L^2(0,T;W^{1,2}(\Omega_j)^3) \text{ and strongly in } L^2(0,T;L^2(\Omega_j)^3); \\ (\nabla w^{(k)}_{(j)})_{k=1}^\infty \text{ converges to } \nabla w_{(j)} \text{ weakly in } L^2(0,T;L^2(\Omega_j)^{3\times 3}); \\ (w^{(k)}_{(j)}(t))_{k=1}^\infty \text{ converges to } w_{(j)}(t) \text{ strongly in } L^2(\Omega_j)^3 \text{ for all } t \in [0,T)\backslash \mathcal{N}_j. \end{cases}$$

First let $j = 1$. Then we use Lemma 4.4, b), with Ω_0 replaced by Ω_1, and find in the same way as in (4.79) a subsequence $(w^{(k)}_{(1)})_{k=1}^\infty$ of $(w^{(k)})_{k=1}^\infty$, a function $w_{(1)} \in L^2(0, T; W^{1,2}(\Omega_1)^n)$, and a null set $\mathcal{N}_1 \subseteq [0, T)$, such that (4.102) holds with $j = 1$. Then we choose $(w^{(k)}_{(2)})_{k=1}^\infty$ as a subsequence of $(w^{(k)}_{(1)})_{k=1}^\infty$, and we choose $w_{(2)} \in L^2(0, T; W^{1,2}(\Omega_2)^n), \mathcal{N}_2 \subseteq [0, T)$, such that (4.102) is satisfied with $j = 2$. Next we choose $(w^{(k)}_{(3)})_{k=1}^\infty$ as a subsequence of $(w^{(k)}_{(2)})_{k=1}^\infty$ satisfying (4.102) together with some $w_{(3)}, \mathcal{N}_3$, for $j = 3$, and so on.

Thus we get a sequence $(w^{(k)}_{(j)})_{k,j=1}^\infty$ of subsequences of $(w^{(k)})_{k=1}^\infty$ which we can write as the lines of a matrix. Then we take the diagonal sequence, which is a subsequence of $(w^{(k)})_{k=1}^\infty$ and satisfies (4.102) simultaneously for all $j \in \mathbb{N}$. For simplicity we may assume that $(w^{(k)})_{k=1}^\infty$ itself has this property.

This construction shows that for all $j \in \mathbb{N}$, $w_{(j)}$ is the restriction of $w_{(j+1)}$ to $[0, T) \times \Omega_j$. Thus we get a well defined function $w \in L^2(0, T; W^{1,2}_{loc}(\Omega)^n)$, so that $w_{(j)}$ coincides with the restriction of w to $[0, T) \times \Omega_j, j \in \mathbb{N}$.

Noting $(w^{(k)})_{k=1}^\infty$ is a bounded sequence in the Hilbert space $L^2(0, T; W^{1,2}_{0,\sigma}(\Omega))$ and we obtain a weakly convergent subsequence; we may assume that the sequence itself has this property. Now the above convergence properties in particular show that $(w^{(k)})_{k=1}^\infty$ converges weakly in $L^2(0, T; W^{1,2}_{0,\sigma}(\Omega))$ to $w \in L^2(0, T; W^{1,2}_{0,\sigma}(\Omega))$. Further we conclude from (4.77) that

$$w \in L^\infty(0, T; L^2_\sigma(\Omega)) \cap L^2(0, T; W^{1,2}_{0,\sigma}(\Omega)).$$

Let the null set $\mathcal{N} \subseteq [0, T)$ be defined as the union of all $\mathcal{N}_j, j \in \mathbb{N}$. From (4.102), we conclude

$$\lim_{k \to +\infty} ||w^{(k)}(t) - w(t)||_{L^2(\Omega_j)^3} = 0, \tag{4.103}$$

for all $t \in [0, T) - \mathcal{N}$ and for all $j \in \mathbb{N}$. In particular, we can conclude that $(w^{(k)}(t))_{k=1}^{\infty}$ tends to $w(t)$ weakly in $L^2_\sigma(\Omega)$, for all $t \in [0, T) - \mathcal{N}$. Moreover, we can get

$$||w(t)||_{L^2(\Omega)^3} \leq \liminf_{k \to +\infty} ||w^{(k)}(t)||_{L^2(\Omega)^3}, \tag{4.104}$$

for all $t \in [0, T) - \mathcal{N}$.

Consider now any test function $v \in C_0^\infty([0, T); C^\infty_{0,\sigma}(\Omega))$. Then we can choose some $j \in \mathbb{N}$ with

$$\text{supp } v \subseteq [0, T) \times \Omega_j.$$

To prove (4.92) we may now use the same convergence properties as in the above case for bounded Ω, T. This yields (4.92) and shows that w is a weak solution of the Navier-Stokes system (3.11).

To prove the energy inequality (4.100) we use (4.93), (4.94) as above, but we replace the condition (4.95) now by (4.104). This proves (4.100) for all $t \in [0, T) - \mathcal{N}$.

The Lemma 3.5 yields again that w is weakly continuous and b) of Lemma 3.6 is hold, and using (4.101) as above, we see that (4.100) holds for all $t \in [0, T)$. The inequality (3.20) is again a consequence of (4.100). The proof of Lemma 3.6 is complete. □

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