Complex-valued Renyi Entropy

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Abstract

Complex-valued expression models have been widely used in the application of intelligent decision systems. However, there is a lack of entropy to measure the uncertain information of the complex-valued probability distribution. Therefore, how to reasonably measure the uncertain information of the complex-valued probability distribution is a gap to be filled. In this paper, inspired by the Renyi entropy, we propose the Complex-valued Renyi entropy, which can measure uncertain information of the complex-valued probability distribution under the framework of complex numbers, and is also the first time to measure uncertain information in the complex space. The Complex-valued Renyi entropy contains the features of the classical Renyi entropy, i.e., the Complex-valued Renyi Entropy corresponds to different information functions with different parameters q. Meanwhile, the Complex-valued Renyi entropy has some properties, such as non-negativity, monotonicity, etc. Some numerical examples can demonstrate the flexibilities and reasonableness of the Complex-valued Renyi entropy.

Keywords: complex-valued probability distribution, Renyi entropy, complex-valued Renyi entropy, uncertain information

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Preprint submitted to Physica A: Statistical Mechanics and its Applications *May 31, 2021*

1. Introduction

In view of the fact that modeling and measuring uncertain information is the basis of expert and artificial intelligence decision making, researchers have proposed a series of modeling theories, such as probability theory[1], Dempster-Shafer theory[2, 3], fuzzy set theory[4], rough set theory[5] and their extensions[6, 7, 8]. Accordingly, effective theories for measuring uncertain information have also been proposed, such as Shannon entropy[9], belief entropy[10], fuzzy entropy[11], Renyi entropy[12], Tsallis entropy[13] and correlation entropies[11]. These theoretical refinements in real-valued space allow for more practical applications of expert and artificial intelligence decision making, such as group decision-making[14, 15, 16], pattern classification[17, 18], cluster analysis[19, 20], reliability analysis[21, 22] and etc[23, 24, 25, 26].

In recent years, complex-valued functions have been successively introduced into uncertain information modeling theory. Ramot et al extended real-valued affiliation to complex-valued affiliation and proposed complex-valued fuzzy sets[27]. Alkouri and Salleh proposed complex-valued intuitionistic fuzzy sets and discussed the distance measure between complex-valued intuitionistic fuzzy sets[28], and some related theories of complex-valued functions have also been proposed based on belief function[29, 30]. In addition, some information carriers containing complex-valued signals are emerging, such as image signals[31], audio signals[32], physiological signals[33]. However, the current uncertain information measurement methods in the real space space cannot reasonably measure the uncertain information of these complex value theories, further limiting the practical application of complex value theories in many fields. Therefore, proposing some reasonable uncertain information measures in the complex-valued framework is an urgent problem to be addressed.

In this paper, inspired by the Renyi entropy[12], we propose a new entropy for measuring uncertain information in the complex-valued framework, namely the Complex-valued Renyi entropy (CRE). CRE is a generalization of the classical Renyi entropy in complex-valued spaces. With different parameters α, CRE can be transformed into different information functions, and CRE also has some properties, therefore, CRE has a high compatibility, which is proved by some numerical examples. Moreover, several numerical examples explain the reasonableness of CRE.

The remainder of the work is organized as follows. Section 2 introduces the basics of Renyi entropy. The Complex-valued Renyi entropy is presented in Section 3. Section 4 proves the compatibility and reasonableness of Complex-valued Renyi entropy. The work is summarized in Section 5.

2. Preliminaries

In the section, the basic knowledge of Renyi entropy is introduced. In 1970, Renyi proposed the $q-order$ generalized entropy in phase space, called Renyi entropy, which is also is a common generalized entropy, and detailed definitions are as follows:[12]

Definition 2.1 (Renyi entropy)
Suppose that p_j is the probability of falling in the $j-th$ phase frame system state, N is the number of phase frames, and satisfied that $\sum_{j=1}^{N} p_j = 1$, then Renyi entropy is described by the following form.

$$R_q = \frac{1}{1-q} log_2 \sum_{j=1}^{N} (p_j)^q \tag{1}$$

Obviously, the Renyi entropy has some properties in measuring the uncertainty of the probability distribution, defined as follows:

Definition 2.2 (The properties of Renyi entropy)

P.1: Non-negativity, i.e., $R_q \geq 0$.

P.2: Monotonically decreasing.

P.3: Convexity, i.e., if and only if $0 < q \leq 1$, R_q is concave function.

Renyi entropy can be transferred to other information functions as α changes, as described in detail below.

Shannon entropy: when $\alpha = 1$, Renyi entropy degenerates to the Shannon entropy, that is to say, $R_1 = \sum_{j=1}^{N} p_j log_2 p_j$;

Hartley entropy: when $\alpha \to 0$, then $R_{\alpha\to 0} = log_2 N$;

Collision entropy: when $\alpha = 2$, then $R_2 = -log_2 \sum_{j=1}^{N} \left(p_j\right)^2 = -log_2 P(X = Y)$, where X and Y are independently and identically distributed;

3. The Complex-valued Renyi entropy complex-valued probability distribution

In this section, we propose the Complex-valued Renyi entropy (CRE) to measuring uncertain information in the complex-valued space, it can be converted to other information functions, being defined as follows:

Definition 2.2 (Complex-valued Renyi entropy)

Suppose there is a complex-valued probability distribution $P = \left\{\hat{p_1}, \hat{p_2}, \cdots, \hat{p_j}, \cdots \hat{p_n}\right\}$, where $\hat{p_j} = p_j e^{i2\pi\theta_j}$ and satisfied that $\sum_{j=1}^{n} \left(p_j\right)^2 = 1, \theta_j \in [0, 1]$. The uncertainty of the complex-valued probability distribution is measured by the following format.

$$\check{R}_q = \frac{1}{1-q} log_2 \sum_{j=1}^{n} \left(\hat{p_j}\hat{p_j}'\right)^q \tag{2}$$

where $\hat{p_j}'$ is the conjugate of $\hat{p_j}$. Next, we discuss the information function corresponding to Complex-valued Renyi entropy when α changes.

- when $\alpha = 1$, CRE degraded to Shannon entropy of complex-valued framework.

  Consider

$$\frac{1}{1-q} log_2 \sum_{j=1}^{N} \left(\hat{p_j}\hat{p_j}'\right)^q = \frac{1}{-1} \frac{\sum_{j=1}^{n} \left(\hat{p_j}\hat{p_j}'\right)^q ln\left(\hat{p_j}\hat{p_j}'\right)}{\sum_{j=1}^{n} \left(\hat{p_j}\hat{p_j}'\right)^q} = -\sum_{j=1}^{n} \hat{p_j}\hat{p_j}' ln\left(\hat{p_j}\hat{p_j}'\right)$$

Next,

$$-\sum_{j=1}^{n}\hat{p}_j\hat{p}_j{}'ln\left(\hat{p}_j\hat{p}_j{}'\right)=-\left(\sum_{j=1}^{n}\hat{p}_j\hat{p}_j{}'ln\hat{p}_j+\sum_{j=1}^{n}\hat{p}_j\hat{p}_j{}'ln\hat{p}_j{}'\right)$$

$$=-\left[\left(\sum_{j=1}^{n}\hat{p}_j\hat{p}_j{}'lnp_j+\sum_{j=1}^{n}\hat{p}_j\hat{p}_j{}'lne^{i2\pi\theta_j}\right)+\left(\sum_{j=1}^{n}\hat{p}_j\hat{p}_j{}'lnp_j+\sum_{j=1}^{n}\hat{p}_j\hat{p}_j{}'lne^{-i2\pi\theta_j}\right)\right]$$

$$=-\left(\sum_{j=1}^{n}\hat{p}_j\hat{p}_j{}'lnp_j+i\sum_{j=1}^{n}\hat{p}_j\hat{p}_j{}'2\pi\theta_j\right)-\left(\sum_{j=1}^{n}\hat{p}_j\hat{p}_j{}'lnp_j-i\sum_{j=1}^{n}\hat{p}_j\hat{p}_j{}'2\pi\theta_j\right)$$

$$\propto-\left(\sum_{j=1}^{n}\hat{p}_j\hat{p}_j{}'log_2p_j+i\sum_{j=1}^{n}\hat{p}_j\hat{p}_j{}'2\pi\theta_j\right)-\left(\sum_{j=1}^{n}\hat{p}_j\hat{p}_j{}'log_2p_j-i\sum_{j=1}^{n}\hat{p}_j\hat{p}_j{}'2\pi\theta_j\right)$$

Therefore, we describe $\breve{R}_1$ as follows:

$$\breve{R}_1=\left\|-2\left(\sum_{j=1}^{n}\hat{p}_j\hat{p}_j{}'log_2p_j-\frac{i\sum_{j=1}^{n}\hat{p}_j\hat{p}_j{}'2\pi\theta_j}{2\pi}\right)\right\| \tag{3}$$

- when $\alpha\rightarrow 0$, CRE degraded to Hartley entropy of complex-valued framework.

$$\breve{R}_{\alpha\rightarrow 0}=\frac{1}{1-0}log_2\sum_{j=1}^{n}\left(\hat{p}_j\hat{p}_j{}'\right)^0=log_2N \tag{4}$$

- when $\alpha=2$, CRE degraded to Collision entropy of complex-valued framework.

$$\breve{R}_2=\frac{1}{1-2}log_2\sum_{j=1}^{n}\left(\hat{p}_j\hat{p}_j{}'\right)^2=-log_2\sum_{j=1}^{n}\left(\hat{p}_j\hat{p}_j{}'\right)^2 \tag{5}$$

Next, we discuss the properties of the CRE.

P.1: Non-negativity, i.e., $\breve{R}_q\geq 0$.

Proof:

Consider $q > 1$, and $0 \leq \hat{p}_j\hat{p}_j{}' \leq 1$, then $\sum_{j=1}^{n}\left(\hat{p}_j\hat{p}_j{}'\right)^q \leq \sum_{j=1}^{n}\hat{p}_j\hat{p}_j{}' = 1$. Further, $log_2\sum_{j=1}^{n}\left(\hat{p}_j\hat{p}_j{}'\right)^q \leq 0$, finally, $\breve{R}_q \geq 0$.

When $q < 1$, $\sum_{j=1}^{n}\left(\hat{p}_j\hat{p}_j{}'\right)^q \geq \sum_{j=1}^{n}\hat{p}_j\hat{p}_j{}' = 1$. Then, $log_2\sum_{j=1}^{n}\left(\hat{p}_j\hat{p}_j{}'\right)^q \geq 0$, and $\breve{R}_q \geq 0$.

When $q = 1$, CRE degenerates to $\breve{R}_1$, and $\breve{R}_1 \geq 0$, that is to say, $\breve{R}_q \geq 0$.

P.2: Monotonically decreasing, i.e., $\breve{R}_q$ decreases as q increases.

Proof:

By taking the derivative of $\breve{R}_q$ with respect to parameter q, we can obtain:

$$\frac{d\breve{R}_q}{dq} = \frac{1}{1-q}\sum_{i=1}^{n}\frac{1}{\left(\hat{p}_j\hat{p}_j{}'\right)}ln\left(\hat{p}_j\hat{p}_j{}'\right) + \frac{1}{(1-q)^2}ln\sum_{i=1}^{n}\left(\hat{p}_j\hat{p}_j{}'\right)^q$$

$$= \frac{1}{1-q}\sum_{j=1}^{n}\frac{\left(\hat{p}_j\hat{p}_j{}'\right)^q}{\sum_{j=1}^{n}\left(\hat{p}_j\hat{p}_j{}'\right)^q}ln\hat{p}_j\hat{p}_j{}' - \frac{1}{(1-q)^2}ln\frac{1}{\sum_{i=1}^{n}\left(\hat{p}_j\hat{p}_j{}'\right)^q}$$

$$= \frac{1}{1-q}\sum_{j=1}^{n}\frac{\left(\hat{p}_j\hat{p}_j{}'\right)^q}{\sum_{j=1}^{n}\left(\hat{p}_j\hat{p}_j{}'\right)^q}ln\hat{p}_j\hat{p}_j{}' - \frac{1}{(1-q)^2}ln\frac{\sum_{i=1}^{n}\left(\hat{p}_j\hat{p}_j{}'\right)}{\sum_{i=1}^{n}\left(\hat{p}_j\hat{p}_j{}'\right)^q}$$

$$= \frac{1}{1-q}\sum_{j=1}^{n}\frac{\left(\hat{p}_j\hat{p}_j{}'\right)^q}{\sum_{j=1}^{n}\left(\hat{p}_j\hat{p}_j{}'\right)^q}ln\hat{p}_j\hat{p}_j{}' - \frac{1}{(1-q)^2}ln\frac{\sum_{i=1}^{n}\left(\hat{p}_j\hat{p}_j{}'\right)^q}{\sum_{i=1}^{n}\left(\hat{p}_j\hat{p}_j{}'\right)^q}\left(\hat{p}_j\hat{p}_j{}'\right)^{1-q}$$

$$= \frac{1}{(1-q)^2}\left[\sum_{j=1}^{n}\frac{\left(\hat{p}_j\hat{p}_j{}'\right)^q}{\sum_{j=1}^{n}\left(\hat{p}_j\hat{p}_j{}'\right)^q}ln\left(\hat{p}_j\hat{p}_j{}'\right)^{1-q} - ln\sum_{j=1}^{n}\frac{\left(\hat{p}_j\hat{p}_j{}'\right)^q}{\sum_{j=1}^{n}\left(\hat{p}_j\hat{p}_j{}'\right)^q}\left(\hat{p}_j\hat{p}_j{}'\right)^{1-q}\right]$$

From Jensen's inequality, it follows that if $f(x)$ is a concave function, then $E(f(X)) \leq f(E(X))$, and if $f(x)$ is a convex function, then $E(f(X)) \geq f(E(X))$, and Ln is a concave function, then

$$\sum_{j=1}^{n}\frac{\left(\hat{p}_j\hat{p}_j{}'\right)^q}{\sum_{j=1}^{n}\left(\hat{p}_j\hat{p}_j{}'\right)^q}ln\left(\hat{p}_j\hat{p}_j{}'\right)^{1-q} - ln\sum_{j=1}^{n}\frac{\left(\hat{p}_j\hat{p}_j{}'\right)^q}{\sum_{j=1}^{n}\left(\hat{p}_j\hat{p}_j{}'\right)^q}\left(\hat{p}_j\hat{p}_j{}'\right)^{1-q} \leq 0$$

Thus, $\breve{R}_q$ decreases as q increases.

P.3: Convexity, i.e., if and only if $0 < q \leq 1$, $\breve{R}_q$ is concave function.

Proof:

Consider when $0 < q < 1$, $ln(x)$ and x^q are both concave functions, then $\breve{R}_q = \frac{1}{1-q} log_2 \sum_{j=1}^{n} \left(\hat{p}_j \hat{p}_j{}' \right)^q$ is also a concave functions.

When $q = 1$, $-\left(\sum_{j=1}^{n} \hat{p}_j \hat{p}_j{}' log_2 p_j - \frac{i \sum_{j=1}^{n} \hat{p}_j \hat{p}_j{}' 2\pi\theta_j}{2\pi} \right)$ and $\|\|_2$ are the concave functions, thus, $\breve{R}_1$ is also a concave function.

When $q \geq 1$, $ln(x)$ is a concave function, but x^q is a convex function, then $\breve{R}_q$ is neither a convex function nor a concave function.

In summary, if and only if $0 < q \leq 1$, $\breve{R}_q$ is concave function.

4. Numerical examples

In this section, the flexibilities and reasonableness of CRE for uncertain information measures is explained through several numerical examples in a complex-valued framework.

Example.1 Suppose there exists a set consisting of elementary events, denoted as $E = \{e_1, e_2, e_3\}$, and the complex-valued probability distribution is described in E as follows:

$$\hat{P} = \left\{ 1e^{i2\pi 0}, 0, 0 \right\}$$

With the above equation, we can get to the uncertainty information measurement results corresponding to parameters q, as shown in Table 1. As can be seen in Table 1, for any q, the measurement of the probability distribution $\hat{P}$ is 0, which is an intuitive result, because the probability of occurrence of event e_1 is 1.

Example.2 There exists a complex-valued probability distribution $\hat{P}$ correspond-

Table 1: The measurement of uncertain information in Example 1.

$\breve{R}_q$	$\breve{R}_{q=0}$	$\breve{R}_{q=0.5}$	$\breve{R}_{q\to 1}$	$\breve{R}_{q=2}$	$\breve{R}_{q=8}$
Entropy	0	0	0	0	0

ing to the set of fundamental events $E = \{e_1, e_2, e_3, e_4\}$, noted as follows:

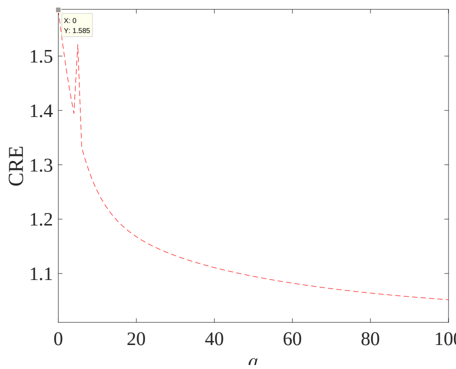
$$\hat{P} = \left\{\sqrt{0.2}e^{i2\pi 0.3}, \sqrt{0.1}e^{i2\pi 0.5}, \sqrt{0.4}e^{i2\pi 0.4}, \sqrt{0.3}e^{i2\pi 0.9}\right\}$$

Table 2: The measurement of uncertain information in Example 2.

$\breve{R}_q$	$\breve{R}_{q=0}$	$\breve{R}_{q=0.5}$	$\breve{R}_{q\to 1}$	$\breve{R}_{q=2}$	$\breve{R}_{q=8}$
Entropy	2	1.9175	2.1391	1.7370	1.4904

The results of CRE measures on $\hat{P}$ are shown in Table 2. It can be seen from Table 2, that different values of p correspond to different results of measure, for example, when $q = 0$, $\breve{R}_0 = 2$, and $q = 0.5$, $\breve{R}_{0.5} = 1.9175$, Obviously, when $q \in [0, 1)$, then $\breve{R}_q$ is monotonically decreasing. When $q = 1$, $\breve{R}_1 = 2.1391$, which characterizes the uncertainty of amplitude and phase angle, and also $\breve{R}_8$ has more uncertainty than $\breve{R}_2$ for $\hat{P}$.

Example.3 Suppose there is a complex-valued probability distribution $\hat{P} = \left\{\sqrt{0.4}e^{i2\pi 0.4}, \sqrt{0.3}e^{i2\pi 0.2}, \sqrt{0.1}e^{i2\pi 0.8}\right\}$ corresponding to the set of elementary events E. when q varies, the uncertainty measure of CRE for this probability distribution is shown in Figure 1. From Figure 1, we can see that $\breve{R}_q$ is larger than 0 when q changes from 0 to 100. Moreover, it proves that $\breve{R}_q$ has monotonic decreasing property.



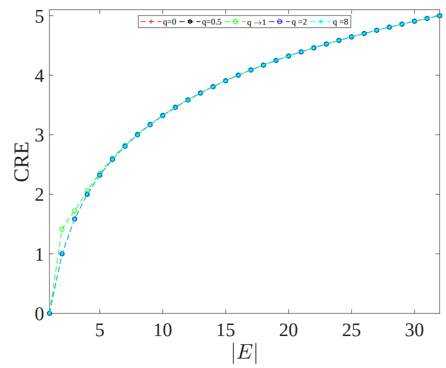


Figure 1: Uncertainty measurement results when q varies

Figure 2: Uncertainty measurement results when $|E|$ varies

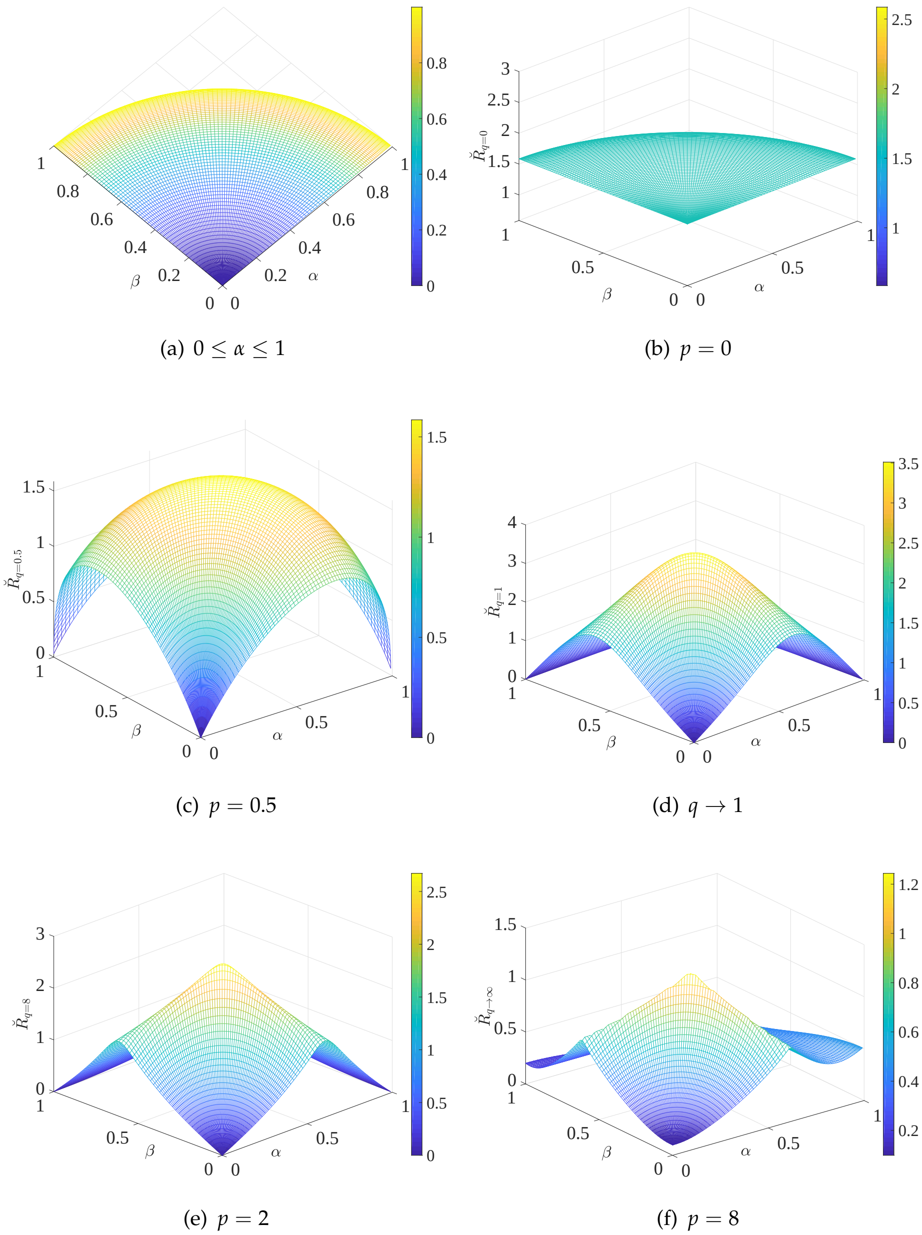
Example.4 Suppose there exists a complex-valued probability distribution $\hat{P}$ in the set of basic events E, the distribution satisfies $\hat{p}_j = \sqrt{\frac{1}{|E|}}e^{i2\pi\frac{1}{|E|}}$, $|E|$ is the number of events, and its variation is shown in Table 3. The uncertainty measure for the complex-valued probability distribution $\hat{P}$ based on CRE is shown in Figure 2. As can be seen from the figure, the uncertainty of this complex-valued distribution increases continuously as the basic events increase. When $|E| = 32$, for any q, $\breve{R}_q$ is the largest. This example demonstrates that CRE can effectively measure the uncertainty of the complex-valued probability distribution under the condition that the number of basic events changes.

Example.5 There is a complex-valued probability distribution $\hat{P} = \{\alpha e^{i2\pi 0.5}, \beta e^{i2\pi 0.2},$

Table 3: The set of basic events E in Example 4.

$\lvert E\rvert$	E
1	$\{e_1\}$
2	$\{e_1, e_2\}$
3	$\{e_1, e_2, e_3\}$
j	$\{e_1, e_2, e_3 \cdots, e_j\}$
32	$\{e_1, e_2, e_3 \cdots, e_j, \cdots, e_{32}\}$

$\sqrt{1-\alpha^2-\beta^2}e^{i2\pi 0.1}\}$ in $E = \{e_1, e_2, e_3\}$, where the relationship between α and β is shown in Figure 3(a). We can see from Figure 3(a) that α and β satisfy $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1$ and $0 \leq \alpha^2 + \beta^2 \leq 1$. Figure 3(b) is the measurement



(a) $0 \leq \alpha \leq 1$ (b) $p = 0$

(c) $p = 0.5$ (d) $q \to 1$

(e) $p = 2$ (f) $p = 8$

Figure 3: The uncertainty measurements based on CRE under different parameters p

of $\hat{P}$ based on CRE when $q = 0$, which shows that $\breve{R}_{q=0}$ does not vary with α and β. This is reasonable. It is know by Equation (4) that the factor affecting $\breve{R}_{q=0}$ is the number of basic events. When $q = 0.5$, the measurement of $\hat{P}$ based on CRE is shown in Figure 3(c). From the figure, we can know that $\breve{R}_{q=0.5} \leq \breve{R}_{q=0}$, regardless of the variation of α and β. When $q \to 1$, $\breve{R}_{q\to 1}$ measures the uncertainty between phase angle and amplitude in $\hat{P}$, as shown in Figure 3(d). Figure 3(d) is asymmetric, which is intuitive. In $\hat{P}$, the phase angles of the three fundamental events are not equal leading to measurements that are also not symmetric. Figure 3(e) and Figure 3(f) show the uncertainty of CRE measurement in $\hat{P}$ when $q = 2$ and $q = 8$, respectively, from which it can be seen that in the same coordinate system, $\breve{R}_{q=2} \geq \breve{R}_{q=8}$. This example demonstrates the reasonable results that can be obtained from the CRE when the support of a proposition changes.

5. Conclusion

In this paper, given that current methods of uncertainty measurement cannot effectively measure the uncertainty of complex-valued probability distributions, inspired by the Renyi entropy, we propose the Complex-valued Renyi entropy. The Complex-valued Renyi entropy provides an effective measure of uncertainty of probability distributions in a complex-valued framework. We discuss the effect of the change of the parameter q in the Complex-valued Renyi entropy, and we also discuss some properties of the Complex-valued Renyi entropy, and further, verify the Complex-valued Renyi entropy can effectively and reasonably measure the uncertainty of the complex-valued probability distribution by numerical examples when the propositional support degree and the number of underlying events change. In addition, there are two future research directions for this work, firstly, first,to study the meaning of the parameter q representation in CRE under the framework of complex-valued probability distribution. Second, some related concepts of the Complex-valued

Renyi entropy, such as divergence and etc, are discussed to further explain its physical meaning and apply it to engineering practice.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgment

The work is partially supported by National Natural Science Foundation of China (Grant No. 61973332), JSPS Invitational Fellowships for Research in Japan (Short-term).

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